Taut foliations and Dehn Surgery along positive braids

Siddhi Krishna (Georgia Tech/Columbia)

December 15, 2020

The L–space Conjecture

Suppose Y is a irreducible $\mathbb{Q}HS^3$. The following are equivalent:

- Y admits a taut foliation "geometry"
- Y is a non-L-space " Floer Homology"
- $\pi_1(Y)$ is left orderable "algebra"

Taut Foliations

Definition

A **taut foliation** is

- a decomposition of Y^3 into codim-1 submanifolds, called **leaves**, such that
- there exists a simple closed curve meeting each leaf transversely

Fibered 3-Manifolds

Start with

- F: a compact, connected, oriented surface
- $\varphi : F \to F$, a diffeomorphism

This determines:

$$M_\varphi = F \times I \big/ \varphi = F \times I \Big/ ((x,0) \sim (\varphi(x),1))$$

make id in a nbhd of $F\times\{0\}$

Taut Foliations in the Literature

Theorem (Thurston '86):
Compact leaves of taut foliations minimize genus in their homology class.

Theorem (Gabai '87):
The "Property R" Conjecture is true, i.e.

$$\text{If } K \subset S^3, \text{ and } S^3_0(K) \approx S^1 \times S^2, \text{ then } K \approx U.$$

Point: L-spaces are "simple" from the Heegaard Floer perspective.

Definition

A closed, irreducible 3-manifold Y is an **L-space** if

$$rk(\widehat{HF}(Y;\mathbb{Z}/2\mathbb{Z})) = |H_1(Y;\mathbb{Z})| < \infty$$

Remark: $rk(\widehat{HF}(Y;\mathbb{Z}/2\mathbb{Z})) \geq |H_1(Y;\mathbb{Z})|$ holds for $\mathbb{Q}HS^3$

Ex: Lens spaces $\subseteq$ Elliptic Mflds

Left-Orderability

Definition

A non-trivial group G is **left–orderable** if there exists a total ordering “ $>$ ” on G that is preserved by left-multiplication.

Non-Example: $\mathbb{Z}/p\mathbb{Z} =< x \mid x^p >$ is not left-orderable.

wlog: assume $x > 1$

$$x^2 > x > 1$$

$$\vdots$$

$$1 = x^p > x^{p-1} > \cdots > x > 1$$

$$\Rightarrow 1 > 1 \text{ ※}$$

Prop: LO groups are torsion free $\square$

The L-Space Conjecture:
Suppose Y is an irreducible $\mathbb{Q}HS^3$.
The following are equivalent:

- Y admits a taut foliation — "geometry"
- Y is a non-L-space — "Heegaard Floer homology"
- $\pi_1(Y)$ is left orderable — "algebra"

Point: These manifolds are "big".

Evidence for the LSC

Theorem (Ozsváth–Szabó; Bowden; Kazez–Roberts):
If Y admits a taut foliation, then Y is a non–L–space.

Theorem: The L-Space Conjecture is true for **graph manifolds**.
Proof: Eisenbud, Hirsch, Neumann, Naimi, Calegari, Walker, Boyer, Gordon, Clay, Watson, Lisca, Stipsicz, J. & S. Rasmussen, Hanselman, and many more!

If Y is an irreducible $\mathbb{Q}HS^3$:

$$\begin{tikzcd} Y \text{ admits a taut foliation} \arrow[r, Rightarrow, bend left, "\text{Ozsváth-Szabó}"] & Y \text{ is a non-L-space} \arrow[l, Rightarrow, bend left, "?"] \end{tikzcd}$$

Questions:

1. How to find non-L-spaces? Dehn surgery.
2. How to build TFs in them?

Producing Non-L-Spaces via Dehn Surgery

Definition

A non-trivial knot $K \subset S^3$ is an **L–space knot** if K there exists $r > 0$ such that $S^3_r(K)$ to an L-space.

Examples: Positive torus knots (Moser) ; Berge Knots

Producing Non-L-Spaces via Dehn Surgery

Suppose $K \subset S^3, K \not\approx U$. Then either:

(1) K is not an L-space knot:

$\longrightarrow$ $m(K)$ is an L–space knot, or

$\longrightarrow$ For all $r \in \mathbb{Q}$, $S^3_r(K)$ is a non-L-space.

(2) K is an L-space knot:

Theorem (Kronheimer-Mrowka-Ozsváth-Szabó; J+S Rasmussen):

$$S^3_r(K) = \begin{cases} \text{non-L-space} & r < 2g(K) - 1 \\ \text{L-space} & r \geq 2g(K) - 1 \end{cases}$$

Suppose $K \subset S^3, K \not\approx U$. Then either:

(1) K **is not an L-space knot:**

$\longrightarrow$ $m(K)$ is an L–space knot, or [Ex: LHT]

$\longrightarrow$ For all $r \in \mathbb{Q}$, $S^3_r(K)$ is a non-L-space. [Ex: 4_1 =

]

(2) K **is an L-space knot:**

Theorem (Kronheimer-Mrowka-Ozsváth-Szabó; J+S Rasmussen):

$$S^3_r(K) = \begin{cases} \text{non-L-space} & r < 2g(K) - 1 \\ \text{L-space} & r \geq 2g(K) - 1 \end{cases}$$

Producing Non-L-Spaces via Dehn Surgery

Suppose $K \subset S^3, K \not\approx U$. Then either:

(1) K **is not an L-space knot:**

$\longrightarrow$ $m(K)$ is an L–space knot, or [Ex: LHT]

$\longrightarrow$ For all $r \in \mathbb{Q}$, $S^3_r(K)$ is a non-L-space. [Ex: 4_1 = ]

(2) K **is an L-space knot:**

Theorem (Kronheimer-Mrowka-Ozsváth-Szabó; J+S Rasmussen):

$$S^3_r(K) = \begin{cases} \text{non-L-space} & r < 2g(K) - 1 \\ \text{L-space} & r \geq 2g(K) - 1 \end{cases}$$

Point: up to the right choice of K or $m(K)$, then $\forall r < 2g(K)-1$, $S^3_r(K)$ is a non-L-space. LSC predicts these ones have TFs.

Relevant Results

$S^3_r(K)$ has a taut foliation if ...

Theorem (Roberts '00)

... K is fibered, $r < 1$

Theorem (K. '20)

... K is a "positive 3–braid closure" and $r < 2g(K) - 1$

Theorem (K.)

... K is a "positive n–braid closure" and $r < \lfloor \frac{4}{3} g(K) \rfloor$

These are the only examples in the literature producing taut foliations in $S^3_r(K)$, $r < f(g(K))$ for hyperbolic knots.

Relevant Results

$S^3_r(K)$ has a taut foliation if . . .

Theorem (Roberts '00)

. . . K is fibered, $r < 1$

Theorem (K. '20)

. . . K is a "positive 3–braid closure" and $r < 2g(K) - 1$

Theorem (K.)

. . . K is a "positive n–braid closure" and $r < \lfloor \frac{4}{3} g(K) \rfloor$

These are the only examples in the literature producing taut foliations in $S^3_r(K), r < f(g(K))$ for hyperbolic knots.

Relevant Results

$S^3_r(K)$ has a taut foliation if ...

Theorem (Roberts '00)

... K is fibered, $r < 1$

Theorem (K. '20)

... K is a "positive 3–braid closure" and $r < 2g(K) - 1$

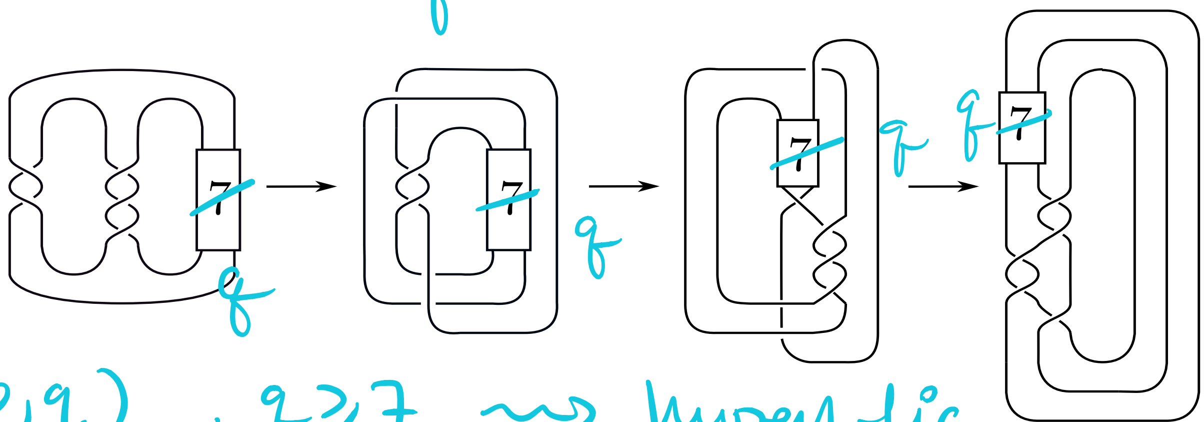
Theorem (K.)

... K is a "positive n–braid closure" and $r < \lfloor \frac{4}{3} g(K) \rfloor$

These are the only examples in the literature producing taut foliations in $S^3_r(K), r < f(g(K)))$ for hyperbolic knots.

Application

Example: The $P(-2,3,\cancel{7})$ q (Fintushel-Stern) Pretzel Knot



$P(-2,3,q)$, $q \geq 7$ $\rightsquigarrow$ hyperbolic

Cor (K. '20)

$$Y \text{ is a non-L-space} \iff Y \text{ admits a taut foliation}$$

holds for **every** non-L-space obtained by Dehn surgery along an infinite family of **hyperbolic** L-space knots.

Cor (K.)

$K_{n,1}$ is braid positive $\iff K \approx U$

"Proof sketch": $(\Leftarrow)$ ✓

$(\Rightarrow)$ By contrapositive. Suppose $K \subset S^3$ and $g(K) \geq 1$. Suppose $K_{n,1}$ is B.P.

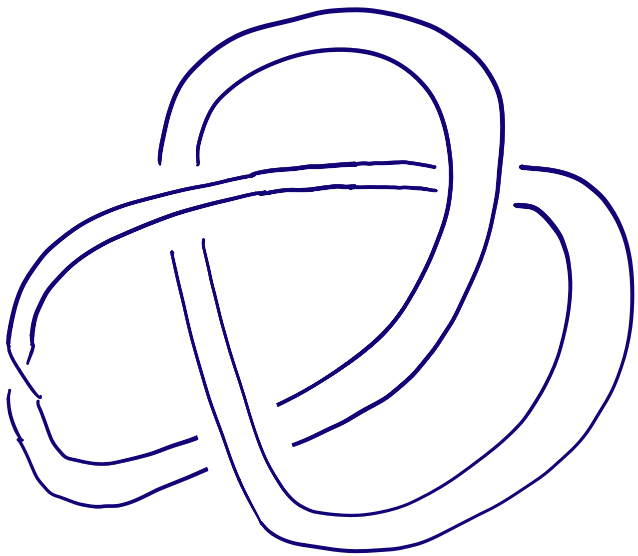
$\forall r < \lfloor \frac{4}{3} g(K_{n,1}) \rfloor$, $S^3_r(K_{n,1})$ has a T.F.

In particular, $S^3_n(K_{n,1})$ has a T.F.

But $S^3_n(K_{n,1})$ is reducible, hence can't have a T.F $\times$

Cor (K.)

If $g(K) \geq 2$, then $K_{n,2}$ is never braid positive.



(2,5) cable of RHT.

Q: If $g(K)=1$, when is $K_{p,q}$ Braid positive?

Q': If K is RHT, when is $K_{p,q}$ Braid Pos?

Challenge: many invts look identical for B.P, Pos, Almost Pos, SQP knots.

For Y an irreducible $\mathbb{Q}HS^3$:

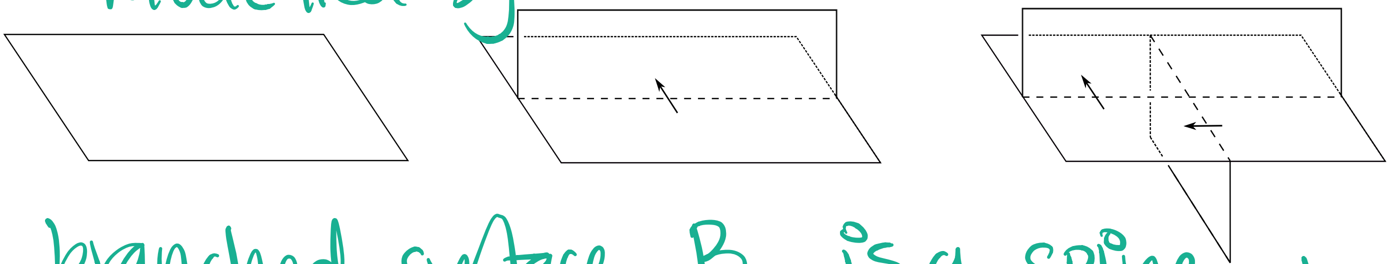
$$\begin{tikzcd}
Y \text{ admits a taut foliation} \arrow[rr, Rightarrow, bend left, "\text{Ozsváth-Szabó}"] & & Y \text{ is a non-L-space} \arrow[ll, Rightarrow, dashed, bend left]
\end{tikzcd}$$

Questions: How do we
(1) identify non-L-spaces? Dehn surgery
(2) build taut foliations in them? Branched surfaces

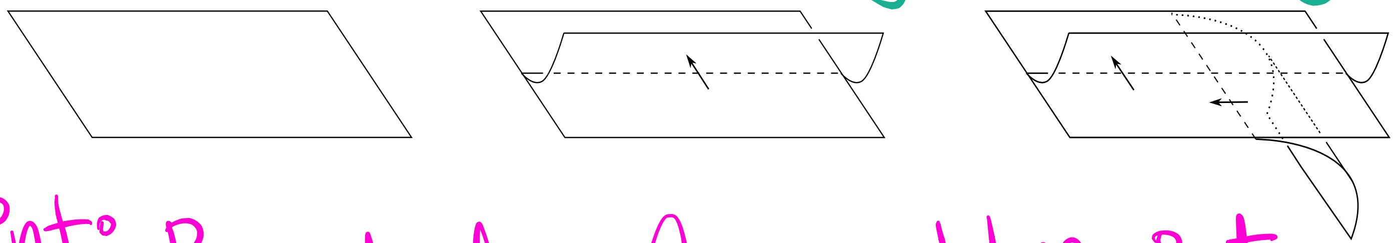
Elements of the construction

- Branched surfaces
- Positive braids
- “Amplification Lemma"

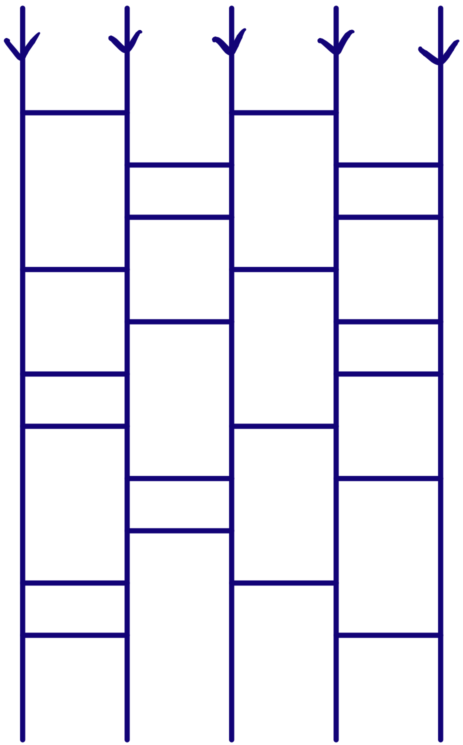
A <u>spine</u> $S \subset Y$ is a 2-complex, locally modelled by:



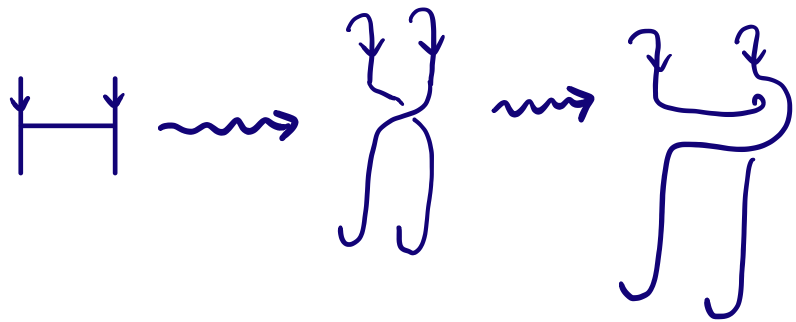
A <u>branched surface</u> B, is a spine + co-orientations, locally modelled by:



<u>Point</u>: Branched surfaces blueprint T.F.s in Y.

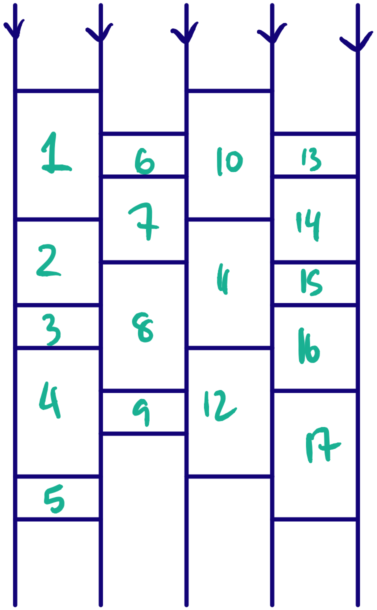


Positive braid: where all crossings are positive

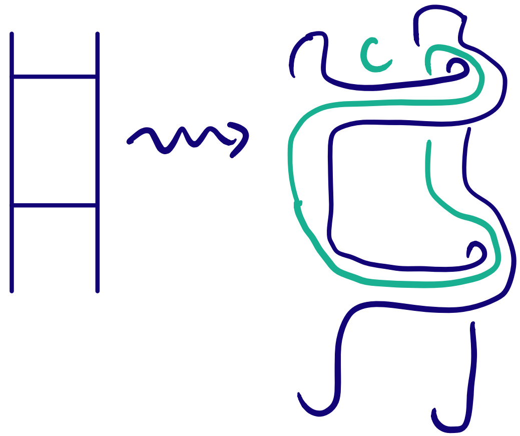


<u>Fact</u> (Stallings)

Pos braid knots are fibered: $X_K = S^3 - \nu(K) = F \times I / \phi$



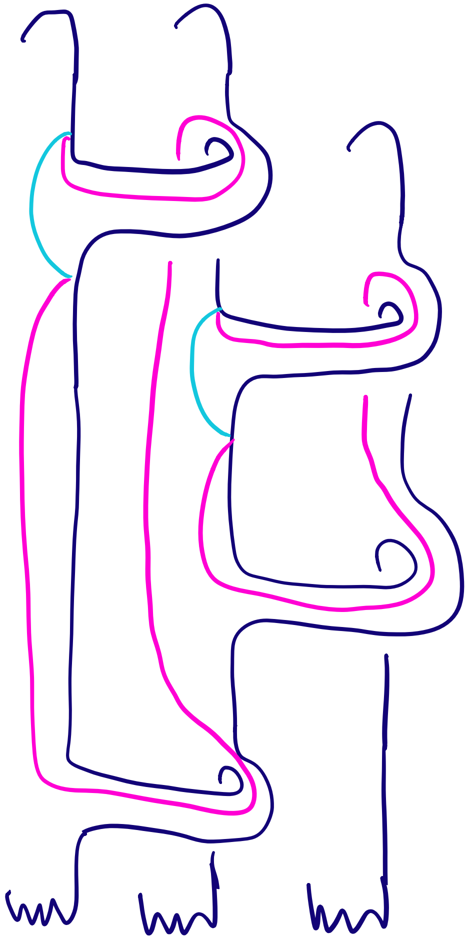
Moreover: Can read off a factorization of Φ directly from the braid word.



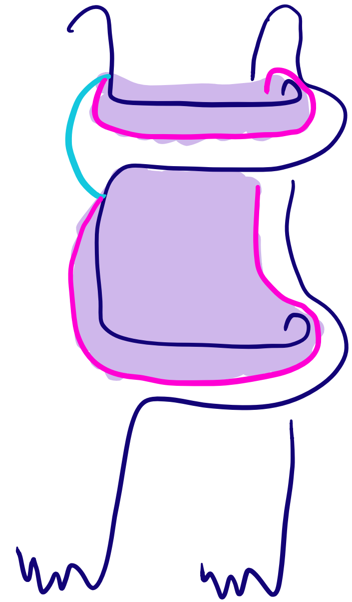
There is a factorization of Φ that includes a pos. Dehn twist about C

$$\Phi = \tau_1 \circ \tau_2 \circ \dots \circ \tau_{16} \circ \tau_{17}$$

Point: Having an explicit factorization of Φ means you can identify $\Phi(\alpha)$, $\alpha \subset F$.



Pushing arcs "through φ" yields disks



"product disk" (à la Gabai)

Point: Build spine S using $F \cup \mathbb{D}^2_s$.

(swept out by plumbing arcs)

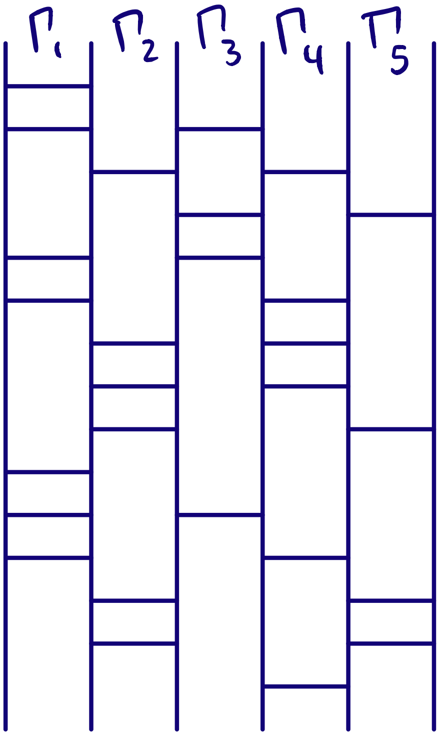
Prop:

- choose k arcs on $F \rightsquigarrow k$ disks $\subset X_K$
- choose right co–orientations for the disks $*$ $*$

Then for all $r < k$, $S^3_r(K)$ has a taut foliation.

Buzzwords: "laminar branched surface"
"sink disk"
"train track"
$\Bigg\}$ Tao Li

Amplification Lemma



$|\Gamma_1| = 7$

$|\Gamma_2| = 6$

$|\Gamma_3| = 4$

$|\Gamma_4| = 6$

$|\Gamma_5| = 4$

① Identify feature in a "local object"

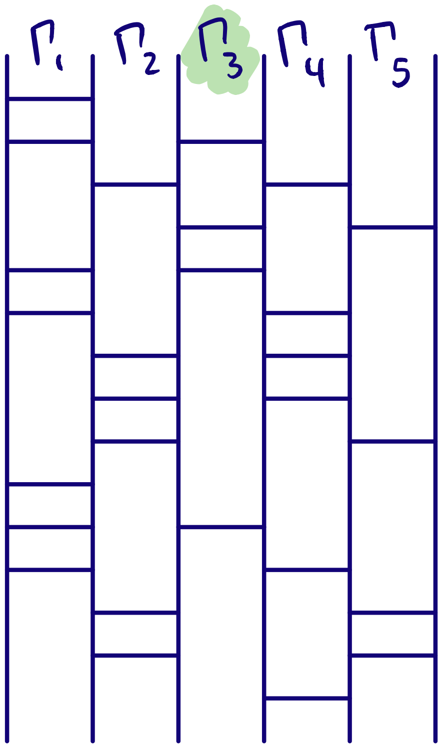
② Find that "local object" many times in your "global object"

Here: Find many pos-3-braids w/in a pos-n-braid.

$$\mathcal{C}_i = \sum_{j \equiv i \bmod 3} |\Gamma_j| \Longrightarrow \begin{array}{l} \mathcal{C}_1 = 13 \\ \mathcal{C}_2 = 10 \\ \mathcal{C}_3 = 4 \end{array}$$

Point: genera of F is "least concentrated" in $\mathcal{C}_3$.

Amplification Lemma



Point:

At most $\frac{c}{3}$ crossings are in Γ_i, $i \equiv 0 \mod 3$

$\Rightarrow$ At least $\frac{2c}{3}$ crossings are Γ_i, $i \not\equiv 0 \mod 3$

$\Rightarrow$ At least $\frac{4}{3}g(K)$ is concentrated in these columns

$\xRightarrow{\text{Prop}}$ $\forall r < \lfloor \frac{4}{3} g(K) \rfloor$, $S^3_r(K)$ has a T.F.