Instanton Floer homology for tangles and applications in Khovanov homology

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(Y, L, ω)

Y closed oriented 3-mfld

$L \subseteq Y$ link

$(\omega, \partial \omega) \subseteq (Y, L)$ 1-mfld

Def (Y, L, ω) is called admissible, if one of the following holds:

(1) $\exists\ \Sigma \subseteq Y$ s.t. $\#\, \Sigma \cap \omega$ odd
$\Sigma \cap L = \emptyset$

(2) $\exists\ \Sigma \subseteq Y$, s.t. $\#\, \Sigma \cap L$ odd

Def (Kronheimer–Mrowka) $I(Y, L, \omega) \cong I(Y, L, [\omega])$

$[\omega] \in H_1(Y, L; \mathbb{Z}/2)$

Example. If L is a knot in S^3.

(S^3, L, ϕ) is not admissible

let m be a meridian of L

u : arc connects L and m

then $I^{\natural}(L) := I(S^3, L \cup m, u)$

Conjecture: $I^{\natural}(L) \overset{?}{\cong} \widehat{HFK}(L)$

(Y, L, ω) admissible

$\Sigma \subseteq Y$ oriented surface, $\Sigma \pitchfork L$.

$\mu(\Sigma) \circlearrowright I(Y, L, \omega)$

$\mu(\Sigma)$: grading $= 2$

$I(Y, L, \omega)$: relatively graded over $\mathbb{Z}/4$

$\mu(pt) \circlearrowright I(Y, L, \omega)$

$\mu(pt)$: grading $= 0$

<u>Prop</u>. If $\Sigma \cap L = \emptyset$

$g(\Sigma) = g$

then the eigenvalues of $\mu(\Sigma)$ on the subspace of $I(Y, L, \omega)$ where $\mu(pt) = 2$, is a subset of $\{-(2g-2), -(2g-4), \dots, (2g-2)\}$.

i.e. $2g-2 \geq |\text{eigenvalues of } \mu(\Sigma)|$

Definition. Suppose $\Sigma \cap L = \emptyset$.

$I(Y, L, \omega \,|\, \Sigma)$

= simultaneous generalized eigenspace

of $(\mu(\Sigma), \mu(pt))$ w/ eigenvalues

$(2g-2, 2)$.

Theorem (Floer, Kronheimer-Mrowka)

If $\Sigma_1, \Sigma_2 \subseteq Y - L$

disjoint. same genus.

$\varphi: \Sigma_1 \to \Sigma_2$ diffeomorphism

$\varphi(\Sigma_1 \cap \omega) = \Sigma_2 \cap \omega$.

$\# \Sigma_i \cap \omega$ odd.

then $I(Y, L, \omega | \Sigma_1) \cap I(Y, L, \omega | \Sigma_2)$

$\cong I(Y', L', \omega' | \Sigma_1') \cap I(Y', L', \omega' | \Sigma_2')$.

Proposition :

If $\Sigma \pitchfork L$

$\# \Sigma \cap L = n$ odd.

$g(\Sigma) = g$.

then the eigenvalues of $\mu(\Sigma)$ in the eigenspace of $\mu(pt)$ w/ eigenvalue 2 is a subset of $\{-2g+2-n, \quad -2g+4-n, \quad \ldots, \quad 2g-2+n\}$

$\Rightarrow \quad |\text{eigenvalue of } \mu(\Sigma)| \leq 2g-2+n$

Def. $\quad I(Y, L, w \mid \Sigma)$

$:=$ simultaneous generalized eigenspace of $(\mu(\Sigma), \mu(pt))$

w/ eigenvalue $(2g-2+n, 2)$

Theorem. (Xie - Z. '19)

The previous excision formula still holds if $\Sigma_i \cap L$ is odd and ≥ 3.

$$\left(\begin{array}{l} \varphi(\Sigma_1 \cap L) = \Sigma_2 \cap L \\ \varphi(\Sigma_1 \cap w) = \Sigma_2 \cap w \end{array} \right).$$

Recall: if $(M, \gamma \subseteq \partial M)$ is a balanced sutured mfld. then Juhász defined a Heegaard Floer homology for (M, γ)

Kronheimer - Mrowka extended it to gauge - ~~theoretic~~ Floer theories using a gluing technique.

__Def__. If (M, γ, T) is a sutured mfld with a tangle. assume, (M, γ) is balanced. We can define $SHI(M, \gamma, T)$ by the strategy of K-M and the singular excision formula.

Prop. (1) If (M, γ, T) is taut, then $SHI(M, \gamma, T) \neq 0$.

(2) If $SHI(M, \gamma, T) \cong \mathbb{C}$ then (M, γ, T) is

$$[0,1] \times (F, \{p_1, \dots, p_n\}).$$

Applications in Khovanov homology

previous detection results:

Kronheimer - Mrowka : unknot

Batson - Seed , Hedden - Ni : unlink

Baldwin - Sivek : trefoil

Baldwin - Sivek - Xie : Hopf link

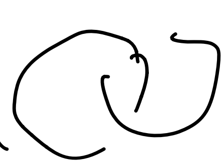
Lipshitz - Sarkar : split link.

Martin : $T(6,2)$

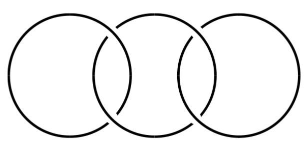
Baldwin - Dowlin - Lidman (?) :

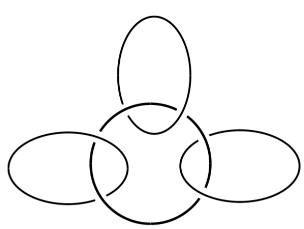
Figure - 8 knot

Remark:
rank $Kh(L; \mathbb{Z}/2) \geq 2^n$
where $L \subseteq S^3$ has n components

$Kh($  $)$ has rank 4.

Def. L is a forest of unlink if L is obtained by connected sums and disjoint unions of Hopf links and unknots.





Theorem (Xie–Z. '19). Let $L \subseteq S^3$ be a link with n components such that

$$\operatorname{rank}_{\mathbb{Z}/2} Kh(L; \mathbb{Z}/2) = 2^n$$

then L is a forest of unknots.

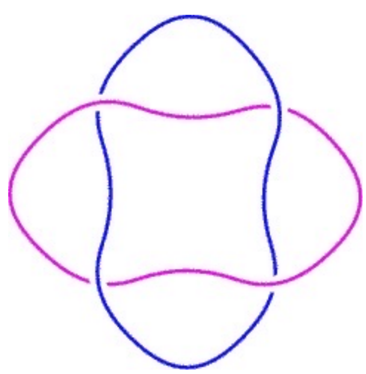
Theorem (Xie - Z. '20)

Suppose $L \subseteq S^3$ is a link with $\operatorname{rank}_{\mathbb{Z}/2} Kh(L; \mathbb{Z}/2) \le 8$, then L is one of the following

(1) forest of unknot with no more than 3 components

(2) trefoil

(3)



Theorem (Xie - Z. '20)

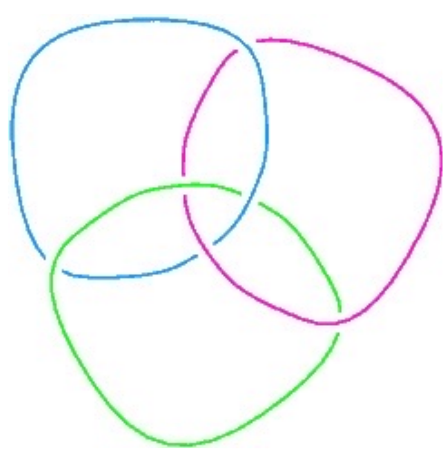
If L is a 3-component link

s.t. $\operatorname{rank}_{\mathbb{Z}/2} Kh(L; \mathbb{Z}/2) \leq 12$

then L is one of the following:

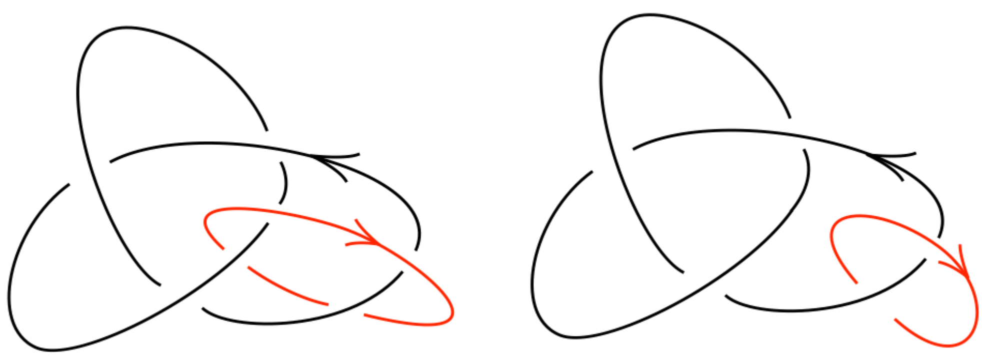
(1) a forest of unknots with 3 components

(2)



Theorem (Li - Xie - Z. '20)

The Khovanov homology distinguishes the following links:



Theorem (Xie - Z. '19)

The annular Khovanov homology detects the trivial link and distinguishes braids from other links.

$L \subseteq D^2 \times S^1$.

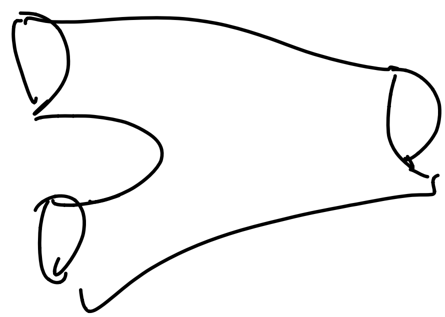
Theorem (Xie - Z. '20)

The Khovanov skein homology defined by Asaeda - Przytycki - Sikora for links in $I \times T^2$ detects knots on T^2.

$L \subseteq I \times D^2 \subseteq I \times T^2$

Proof of the excision formula

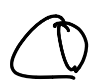
$I(S^1\times\Sigma, \psi, S^1\times pt)$

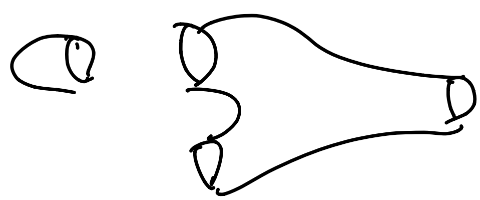
 $\times\ \Sigma$

cobordism from $S^1\times\Sigma \sqcup S^1\times\Sigma \longrightarrow S^1\times\Sigma$.

$$I(S^1\times\Sigma)\otimes I(S^1\times\Sigma)\to I(S^1\times\Sigma)$$

unit element:

 $\times\ \Sigma$



$\cong$ 

Muñoz computed the ring structure.

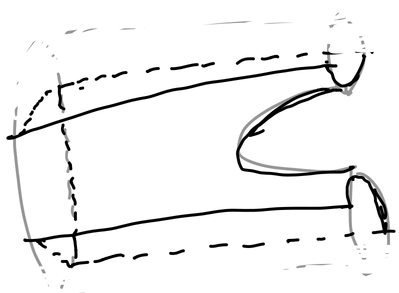
$\mu(\Sigma)(1)$, $\mu(pt)(1)$

$\mu(\gamma_i)(1)$ generate

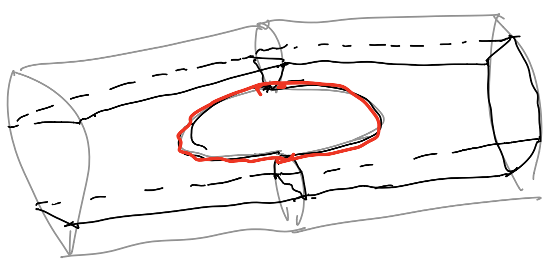
the ring $I(S^1\times\Sigma, \emptyset, S^1\times pt)$

$(\mu(pt)-2)^k = 0$

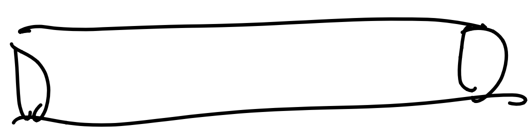
$$\boxed{I(S^1\times\Sigma, \phi, S^1\times pt \mid \Sigma) \cong \mathbb{C}.}$$



$\times\ \Sigma$

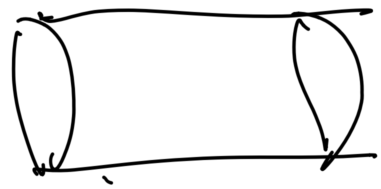


$\times\ \Sigma$ A



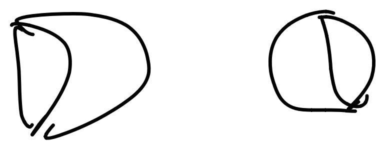
$\times\ \Sigma$ B

A



$\times \Sigma$

B



$\times \Sigma$

$$S^1 \times \Sigma \to S^1 \times \Sigma.$$

Question: How to compute

$$I(S^1 \times \Sigma, \; S^1 \times \{p_1, \dots, p_n\}, \; \phi)$$

as a ring?

difficulty;

(1). algebraic geometry

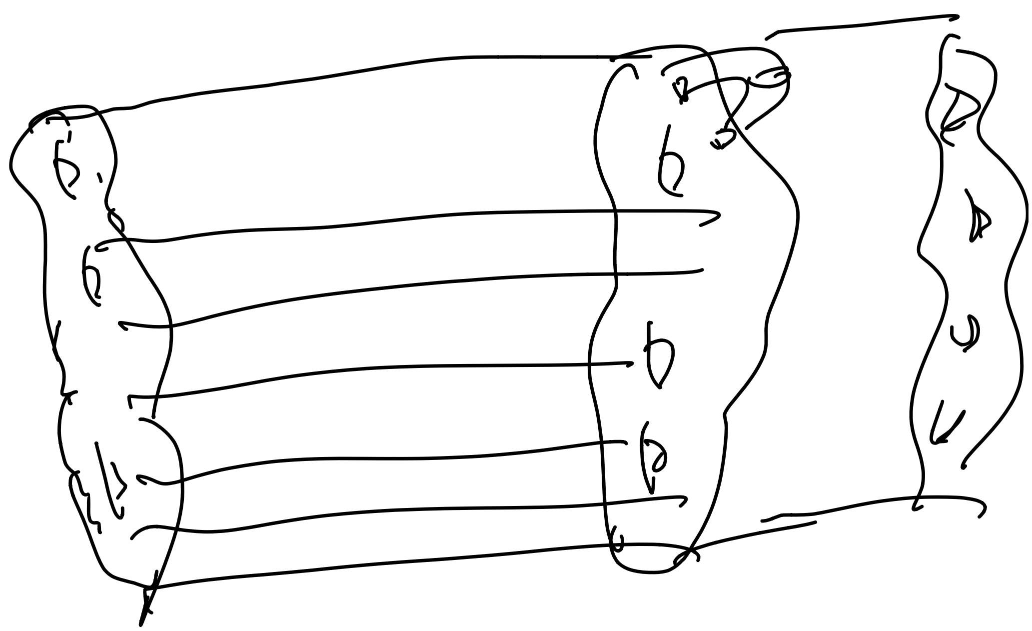
(2) induction argument is difficult.

Our approach:

show $I(S^1 \times \Sigma, S^1 \times \{p_1, \dots, p_n\}, \phi \mid \Sigma) \cong \mathbb{C}$

directly using induction on n and g.

(I)

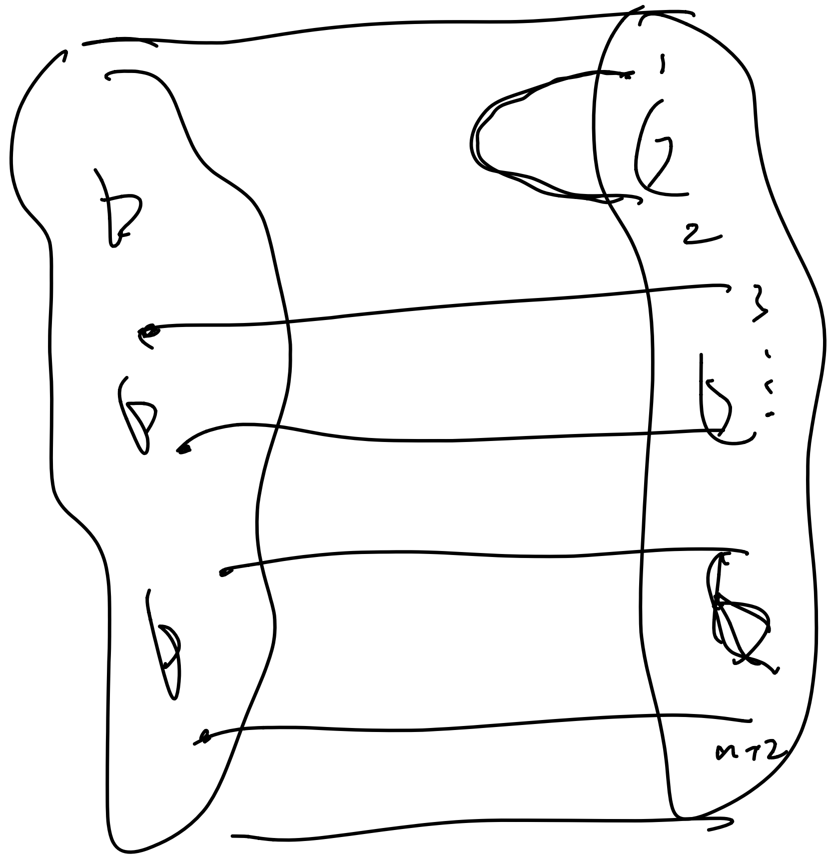
$S^1 \times$ 

injective.

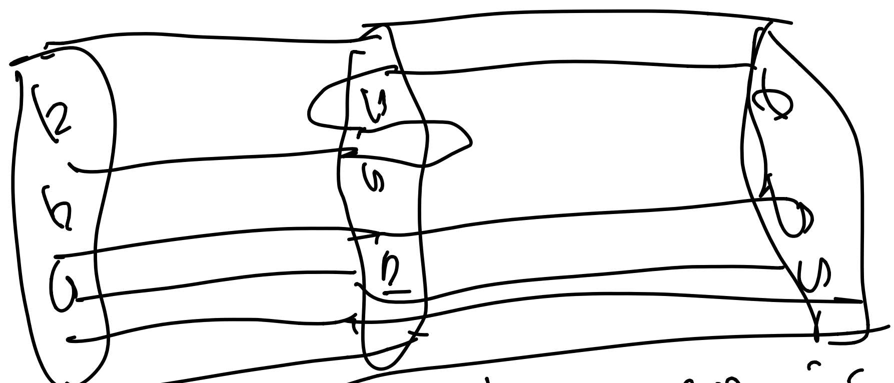
equivariant w.r.t. MCG.

(II)

$S^1 \times$



$$1 \longmapsto (\pm\delta_1, \pm\delta_2)\cdot 1$$



this map is

also injective,

(III) Mumford relation.