Taut Foliations Leafwise branch cover S^2

D. Calegari

Def: Let M^3 be closed, oriented. A co-oriented foliation $\mathcal{F}$ is taut if $\exists$ compact 1-mfld X and a map $X \to M$, $X \pitchfork \mathcal{F}$ which intersects every leaf.

Equivalent definitions:

1. $\exists$ ω closed 2-form $\omega|T\mathcal{F} > 0$
2. $\exists$ metric for which leaves are minimal surfaces
3. $\exists$ volume-preserving $\pitchfork$ flow

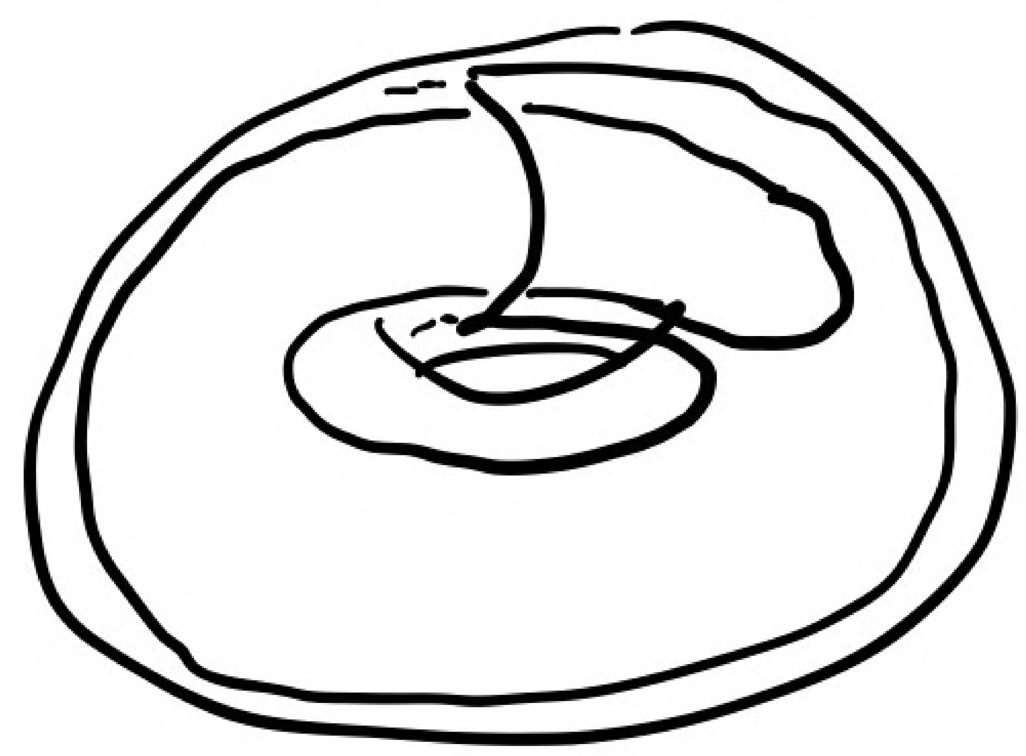
Examples: ∘ surface bundle over S^1

∘ stable/unstable foliation of the geodesic flow on UTS

∘ foliations without compact leaves

∘ examples on every irreducible 3-mfld w/ $H_2 \neq 0$

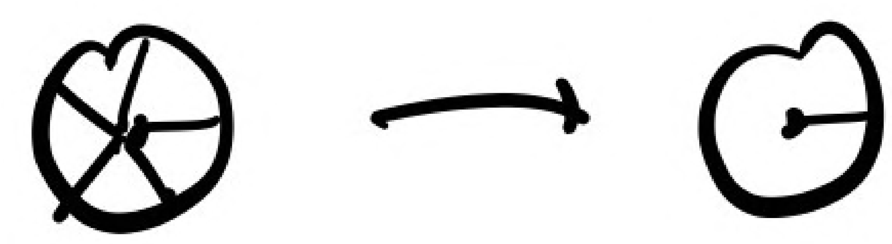
Non-examples: ∘ foliations with Reeb components:



New characterization of tautness:

<u>Thm</u> (C.–) $(M, \mathcal{F})$ is taut if and only if $\exists$ $\phi : M \to S^2$ which is a leafwise (oriented) branched cover; ie if

$\phi : \lambda \to S^2$ looks locally like $z \to z^k$ for some k



By perturbing ϕ suitably we may assume branch points are all <u>simple</u> (ie $k=2$) and branch locus X is a compact 1-mfld $\pitchfork$ to $\mathcal{F}$.

"easy" direction: $\phi: M \to S^2$ with branch locus $X \pitchfork \mathcal{F}$.

Suppose $\exists\ \lambda$ disjoint from X. Then $\phi: \lambda \to S^2$ is an immersion.

Because M is compact, $\phi: \lambda \to S^2$ is a covering map. Thus $\lambda = S^2$.

But if $\mathcal{F}$ has an S^2 leaf, Reeb Stability Theorem $\Rightarrow M = S^2 \times S^1$ and $\mathcal{F}$ = foliation by $S^2 \times pt$. $\square$

This theorem very nearly follows from a Theorem of Ghys that I learned in a lecture at Stony Brook >20 years ago.

The "hard" direction is to show taut $\Rightarrow \exists\ \phi: M \to S^2$.

Thm (Ghys): Let $\mathcal{F}$ be a taut foliation, and suppose the leaves of $\mathcal{F}$ are all hyperbolic. Then $\exists\ \phi: M \to \mathbb{CP}^1$ leafwise holomorphic (and nonconstant).

There is an analytic proof of the "hard" direction:

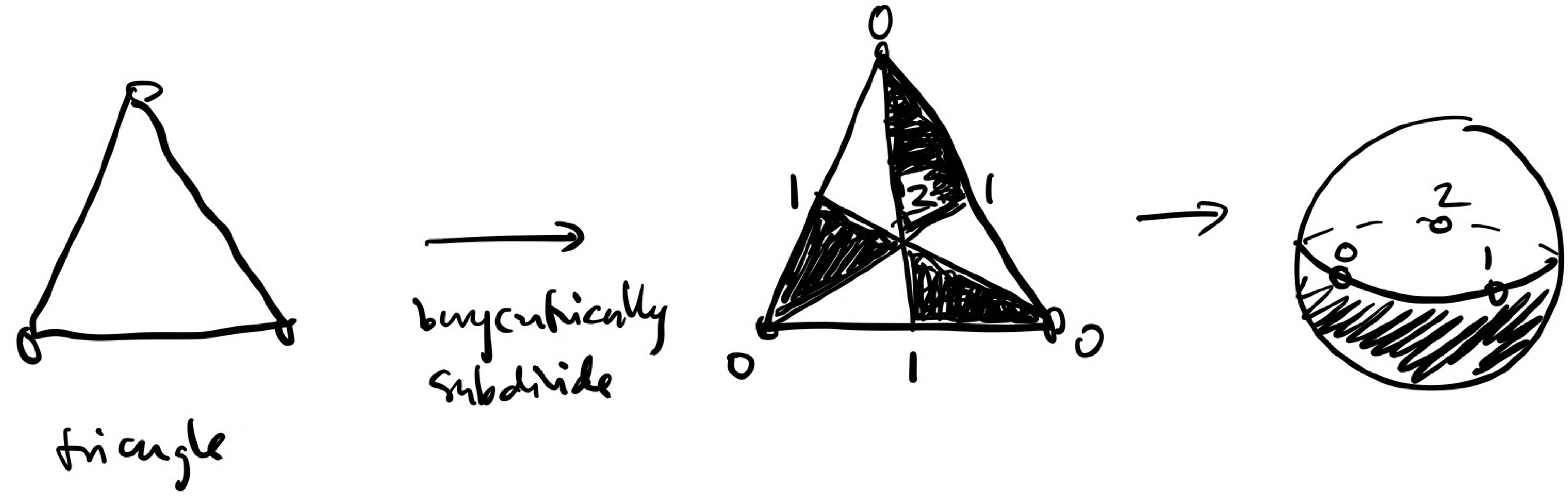
$$\underset{\text{taut}}{(M,\mathcal{F})} \xrightarrow{\text{Eliashberg-Thurston, Bowden, Kazez-Roberts}} \underset{\text{symplectic w/ pseudoconvex }\partial}{(M\times I,\omega)} \xrightarrow{\text{Eliashberg, Etnyre}} \underset{\text{almost-complex symplectic filling}}{(W^4,\omega,J)} \xrightarrow{\text{Donaldson}} \begin{array}{c}(W^4,\omega,J)\\ \phi\downarrow \text{ "Lefschetz pencil"}\\ \mathbb{CP}^1\end{array}$$

restriction to $(M,\mathcal{F})$ is the desired map. $\square$

Remainder of talk: Give a direct combinatorial construction of ϕ (for which one can control other data eg the class in H^2 represented by ϕ).

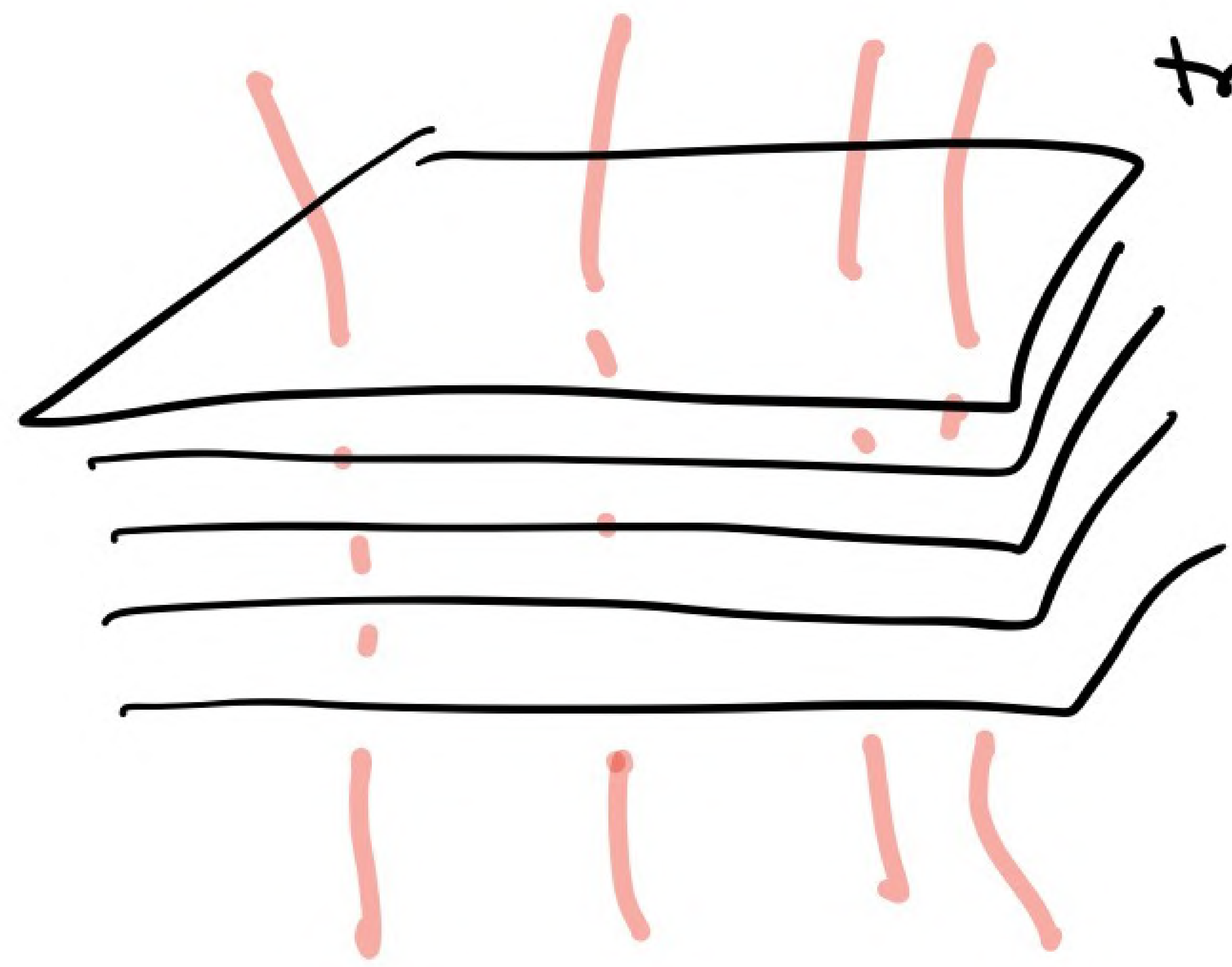
Second construction: from Belyi maps.

S^2 has a triangulation
with 2 triangles:
color white/black



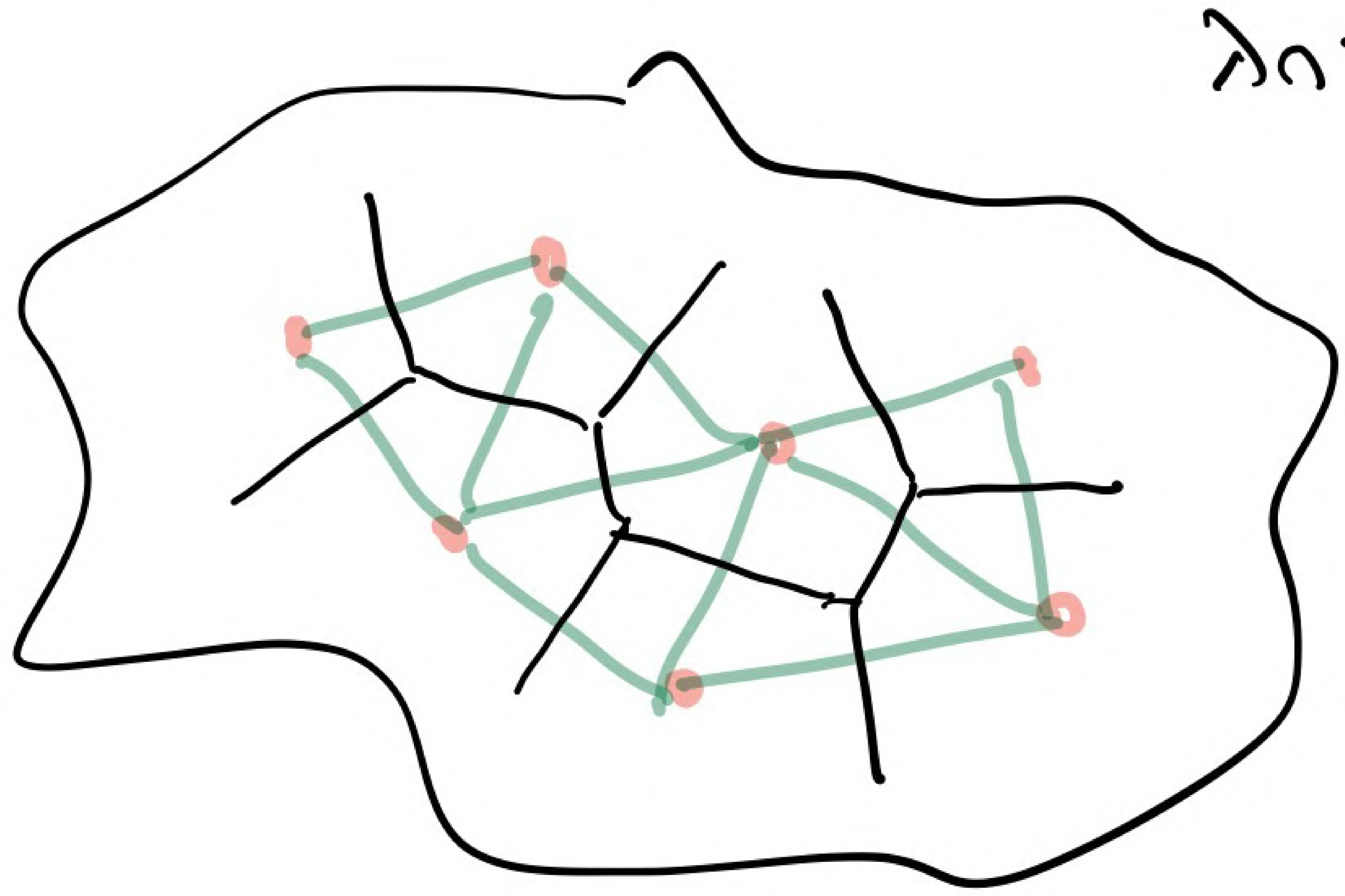
triangle in an oriented surface $\to$ barycentric subdivision gets canonical 2-coloring; vertices labeled by 0, 1, 2 $\to$ map to S^2

$\mathcal{F}$ taut foliation



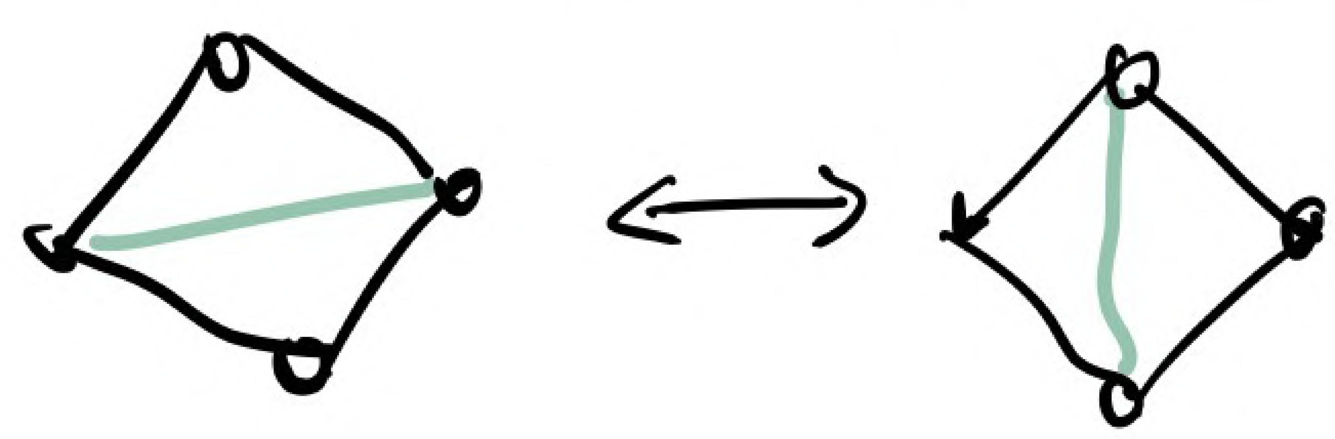
transverse l-fld

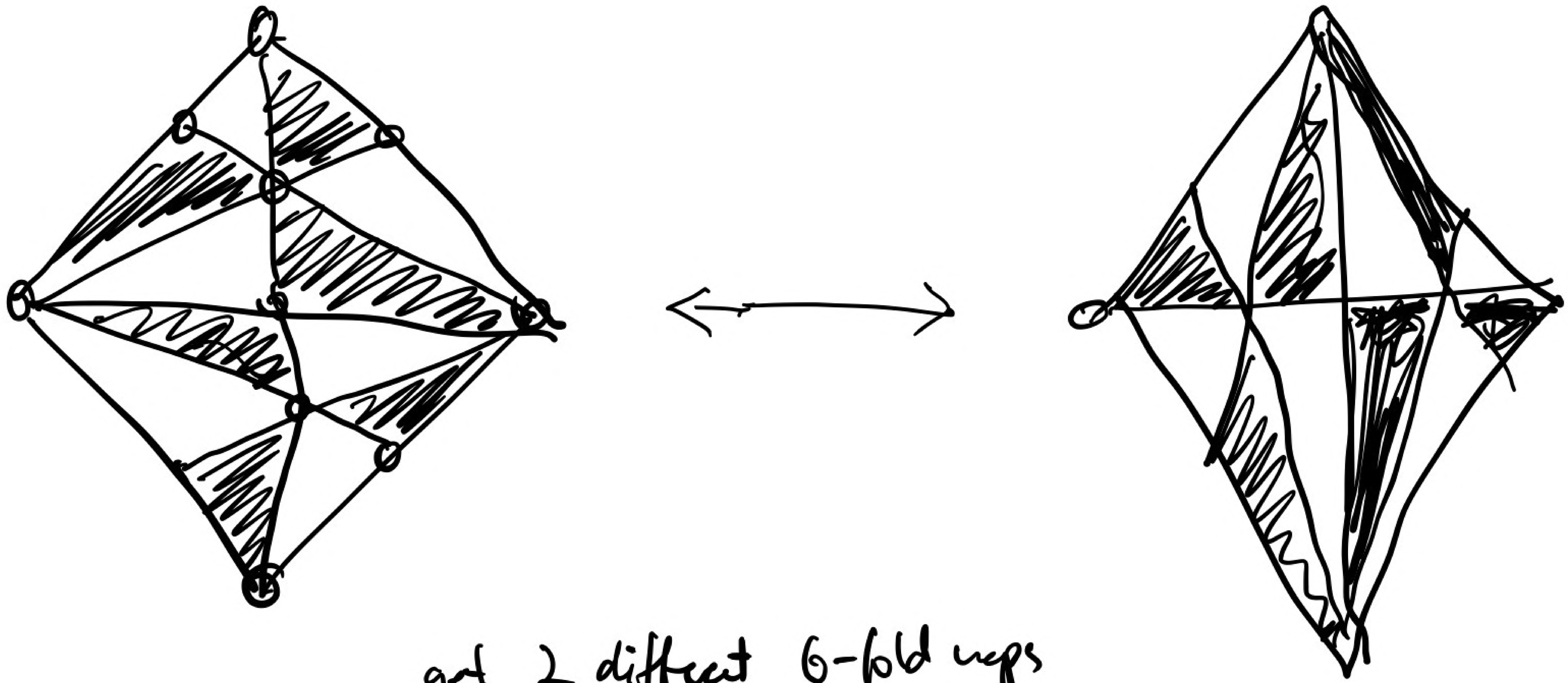
leafwise:



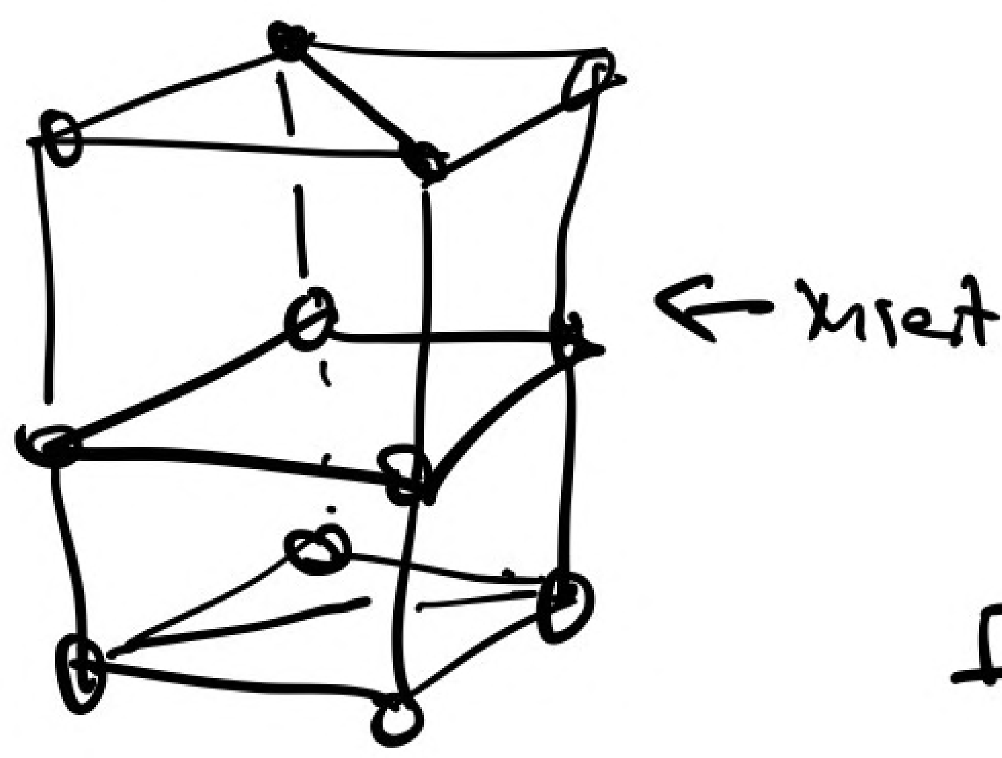
$\lambda \cap$ transversals: collection of points

Voronoi tiling ⇝ dual to triangulation; varies continuously from leaf to leaf except at finitely many isolated places where 2-2 move.





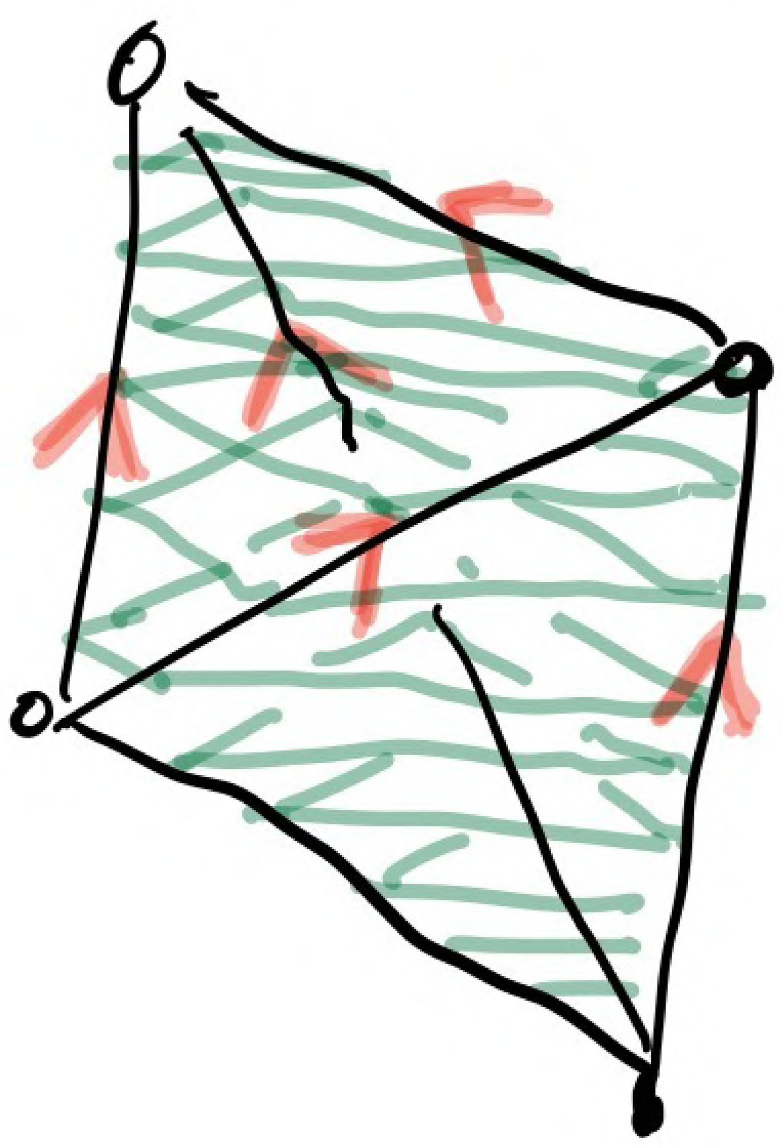
get 2 different 6-fold maps square $\to S^2$ with same boundary values. These maps are homotopic through branched maps; insert this homotopy between the 2 squares:



□

First construction: from triangulations.

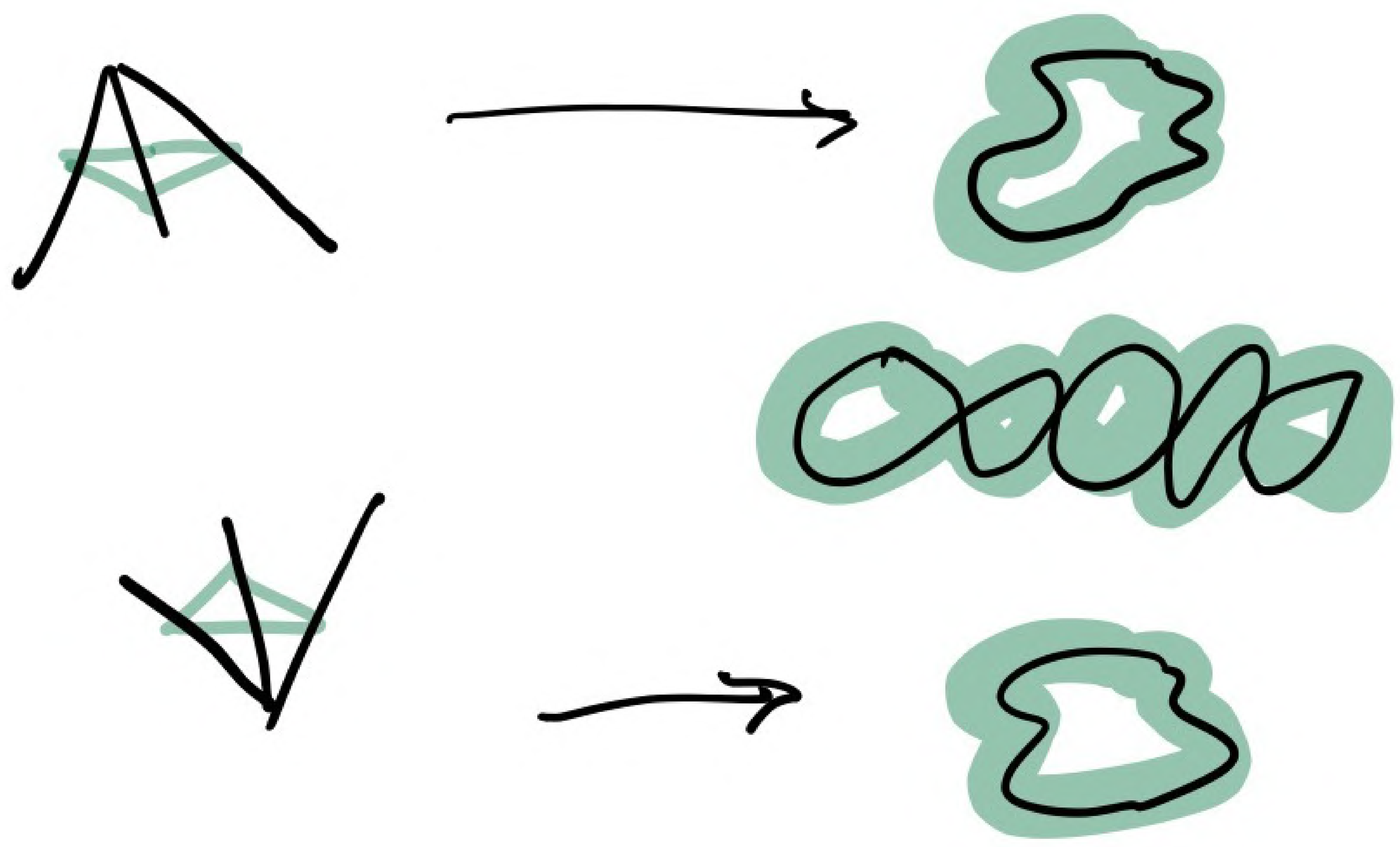
Triangulate M s.t. $\mathcal{F}$ restricts to a "linear" foliation of each simplex:



edges are oriented by coorientation of foliation.

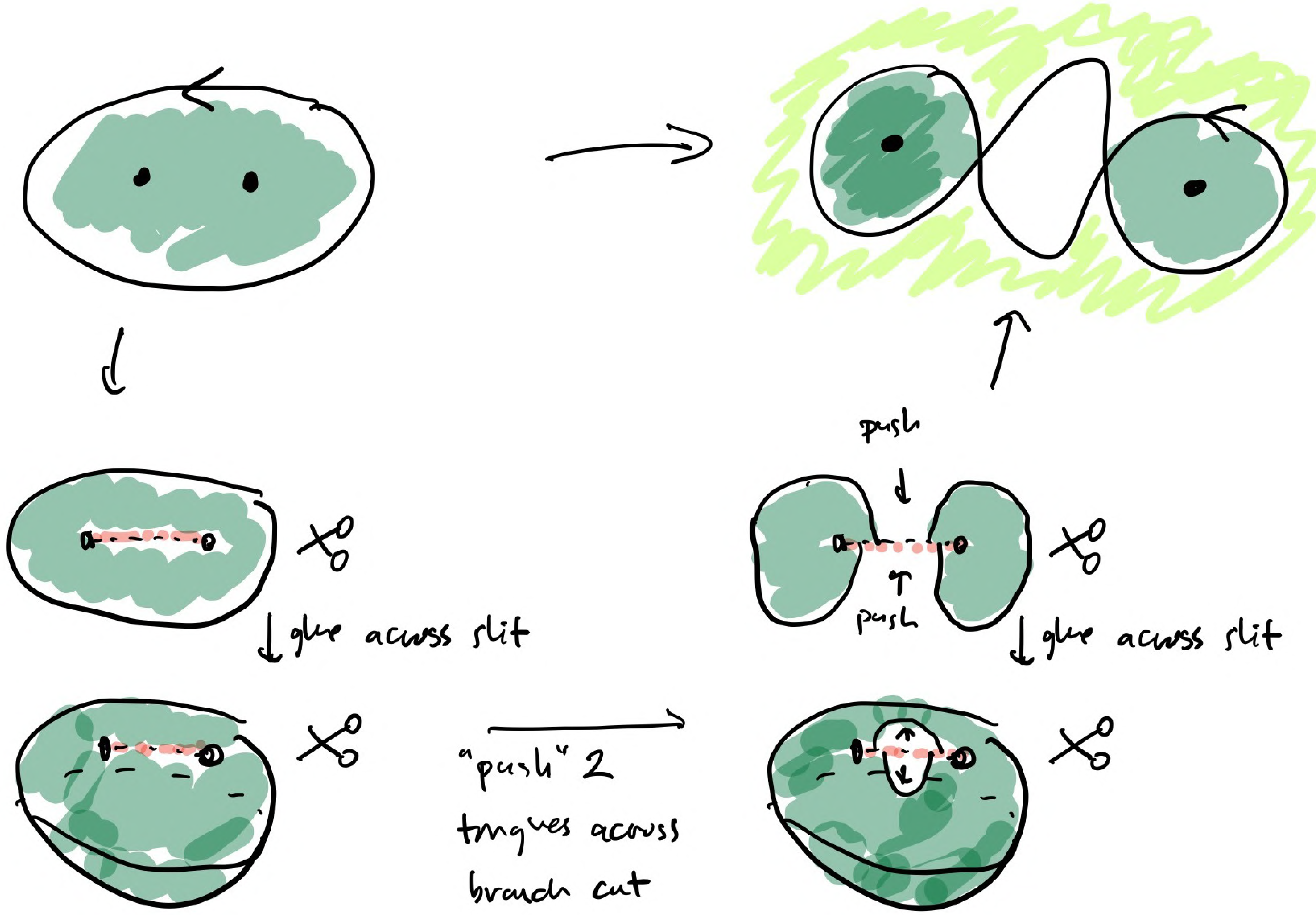
Easy: produce ϕ: Nbhd(2-skeleton) $\to S^2$ which is an immersion on leaf $\cap$ Nbhd(T^2).

Problem: to extend over interior of simplex.



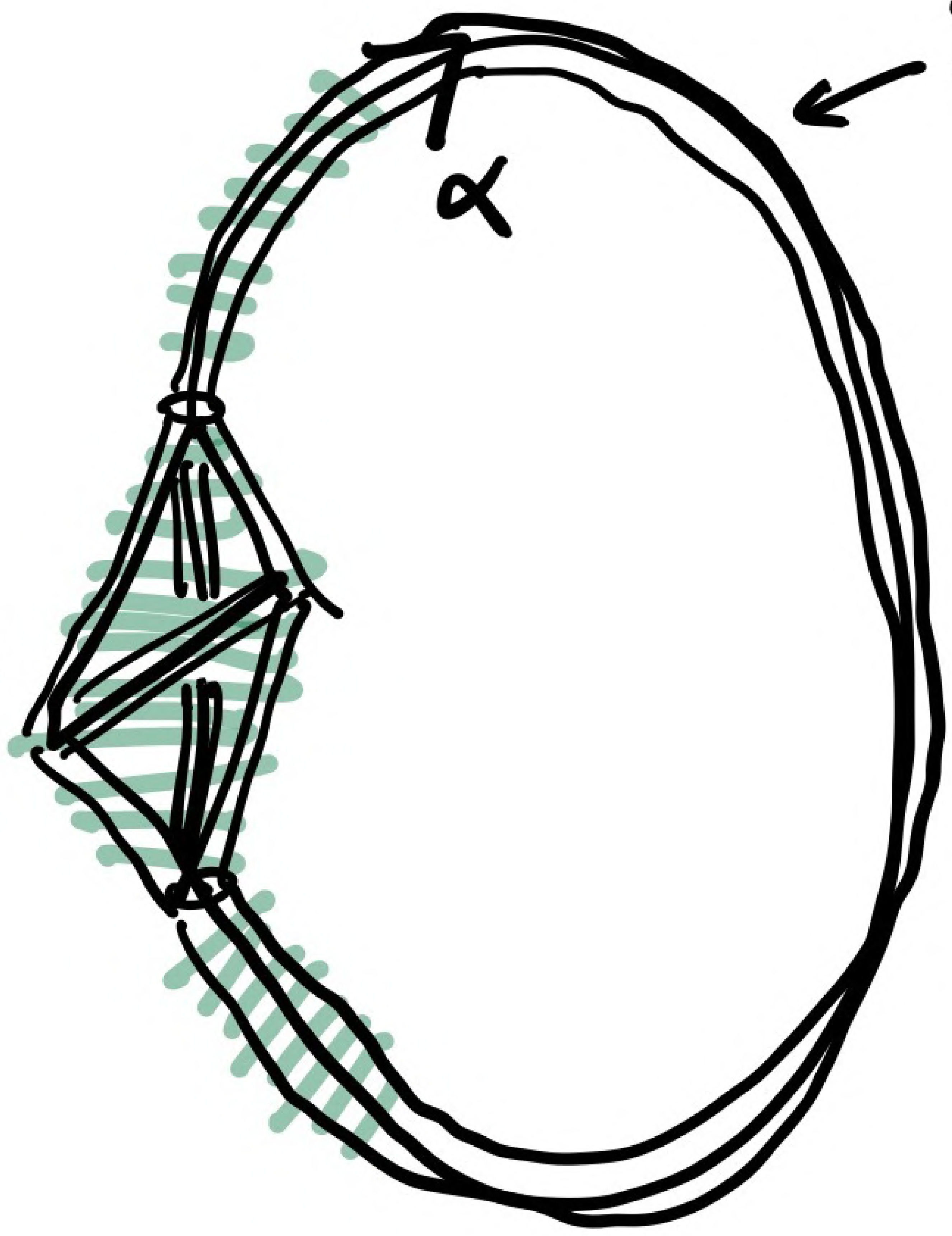
1-parameter family of immersed circles in S^2; need to extend to immersions of disk.

Extension can be done if we allow branch points.


push
push
glue across slit
glue across slit
"push" 2
tongues across
branch cut

simplex

Δ



"drill" arc α out of $\mathrm{Nbhd}(T^2)$ from top to bottom of simplex (uses tautness!)

insert arc of double branch points along α.

then we can solve extension problem in Δ.

$\square$