Knot genus in a fixed 3-manifold

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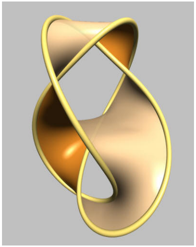
21 July 2020

Knot genus

The genus $g(K)$ of a knot K in the 3-sphere is the minimal genus of a Seifert surface for K.



It's a natural measure of complexity. For example,

$$g(K) = 0 \Leftrightarrow K \text{ is the unknot.}$$

Question: How difficult is it to determine the genus of a knot?

Generalisations

- One can also define the genus of a link L. This link may or may not be oriented.
- One can consider knots K in 3-manifolds M. A knot K in M bounds a compact orientable surface in M if and only if $[K] = 0 \in H_1(M)$.

Computational complexity

Recall that a decision problem is a question that requires a yes/no answer.

For example,

'Given a diagram of a knot K and a natural number g, is $g(K) = g$?'

A decision problem is in P if it can be answered in polynomial time, as a function of the size of the input. In our case, this size is the number of crossings of the diagram plus the number of digits of g.

Non-deterministic polynomial time

A decision problem lies in NP if a ‘yes’ answer can be certified in polynomial time. By certified, we mean that one can be provided with extra data, called a certificate, which can be used to verify that the answer is ‘yes’.

Example: Is a positive integer n composite? This lies in NP because one can certify a ‘yes’ answer by giving two integers $n_1, n_2 > 1$ such that $n_1 n_2 = n$.

- Any NP problem can be solved in exponential time.
- Many problems are NP-complete. This means that if you can solve them, then you can solve any NP problem.

The computational complexity of knot and link genus

Theorem: [Agol-Hass-Thurston, 2002] The problem of determining whether a knot K in a compact orientable 3-manifold M has genus at most g is NP-complete.

Theorem: [L, 2017] The problem of determining whether an unoriented link L in the 3-sphere has genus at most g is NP-complete.

Theorem: [L, 2016, based on Agol 2002] The problem of determining whether a knot K in the 3-sphere has genus at most g is in NP and co-NP.

co-NP

Theorem: [L, 2016, based on Agol 2002] The problem of determining whether a knot K in the 3-sphere has genus at most g is in NP and co-NP.

A decision problem is in co-NP if a 'no' answer can be certified in polynomial time.

Widely believed conjecture : No problem in co-NP is NP-complete.

So, it is almost certainly much easier to deal with knots in the 3-sphere than in general 3-manifolds.

Assuming NP $\neq$ co-NP, the problem of determining whether a knot K in a 3-manifold M has genus equal to g is not in NP.

Knots in a fixed 3-manifold

Theorem: [L-Yazdi, 2020] The problem of determining whether a knot K in a fixed compact orientable 3-manifold M has genus at most g is in NP and co-NP.

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K and M can be given in one of two ways:

- One can fix a surgery diagram of M. Then we give K by adding K to this diagram.
- One can fix a Heegaard diagram of M. Then we give K by drawing a knot diagram in the Heegaard surface.

Thurston norm

For a compact orientable connected surface S,

$$\chi_-(S) = \max\{-\chi(S), 0\}.$$

For a compact orientable surface S with components $S_1, \ldots, S_n$,

$$\chi_-(S) = \sum_i \chi_-(S_i).$$

The Thurston norm of a class $z \in H_2(M, \partial M)$ is

$$x(z) = \min\{\chi_-(S) : S \text{ is a compact oriented surface with } [S] = z\}.$$

Thurston proved that x extends to a semi-norm on $H_2(M, \partial M; \mathbb{R})$.
The unit ball is a polyhedron.

Thurston norm detection lies in NP

Theorem: [L, 2016, based on Agol 2002] The problem of determining the Thurston norm of a homology class lies in NP.

Specifically, one is given:

- a triangulation of a compact orientable 3-manifold M,
- a simplicial 1-cocycle c,
- a natural number n,

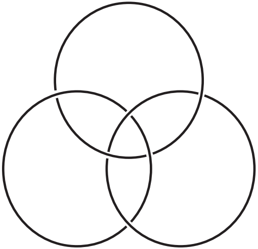
and the decision problem asks whether the Thurston norm of the Poincaré dual of c is n.

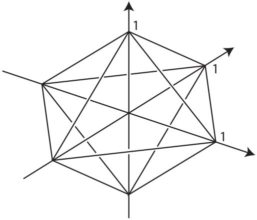
Genus detection using the Thurston norm

1. We have a knot K in a 3-manifold M.
2. Form $X = M \backslash\backslash N(K)$.
3. Consider those classes z in $H_2(X, \partial X)$ such that ∂z is a longitude of K.
4. We want to find the minimal possible $x(z)$.

Computing the norm ball

Thurston showed how to compute the unit ball of the Thurston norm in some specific cases:





Computing the norm ball

A potential certificate:

- a finite list of points V in $H_2(X, \partial X; \mathbb{Q})$ – these will be the vertices;
- a list of subsets $\mathcal{F}$ of V – these will be the faces;
- a certificate for $x(v)$ for each $v \in V$;
- a certificate for $x(w)$ for the barycentre w of each face.

Checking this is challenging. For example, we need to be sure that the faces cover the entire boundary of the unit ball.

Key problem: Why do we have only polynomially many faces (as a function of the number of tetrahedra of X)?

Bounding the number of faces

Theorem: [L-Yazdi, 2020]
Let M be a closed orientable 3-manifold obtained by surgery on a framed link L in S^3.
Let $b = b_1(M)$.
Fix a diagram D for L.
Let K be a knot in M.
Let D' be a diagram for $K \cup L$ with D as a sub-diagram.
Let c be the number of crossings of D'.
Then the Thurston norm ball of $M \backslash\backslash N(K)$ has at most $O(c^{2(b+1)^2})$ faces.

This is a polynomial function of c. The implied constant in $O(\ )$ depends on M, L and D.

Bounding the number of facets

A facet is a top-dimensional face.

We'll show that the number of facets is at most $O(c^{2(b+1)})$.

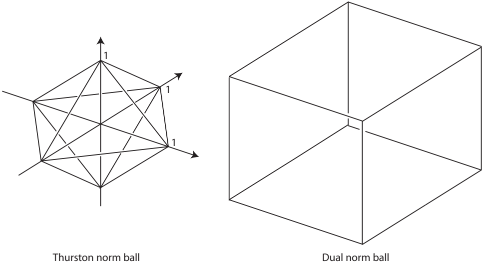
Each face is the intersection of at most $b+1$ facets. So, the number of faces is then at most

$$\binom{O(c^{2(b+1)})}{1} + \binom{O(c^{2(b+1)})}{2} + \cdots + \binom{O(c^{2(b+1)})}{b+1} \leq O(c^{2(b+1)^2}).$$

The dual norm

Given any norm x on a vector space V, there is a dual norm x^* on V^*:

$$x^*(\phi) = \sup\{\phi(v) : x(v) \leq 1\}.$$



Each facet of the norm ball of x corresponds to a vertex of the norm ball of x^*.

Thurston showed that the vertices of x^* are integral, ie elements of $H^2(M, \partial M; \mathbb{Z})$.

Bounding the number of integral points

Theorem: [L-Yazdi, 2020]
Let X be a compact orientable 3-manifold.
Let $S_1, \ldots, S_b$ be a collection of compact oriented surfaces that form a basis for $H_2(X, \partial X; \mathbb{R})$.
Suppose $\chi_-(S_i) \leq m$ for all i.
Then the number of integral points in the unit ball for $H^2(X, \partial X) \otimes \mathbb{R}$ is at most $(2m+1)^b$.

Hence, the number of facets of the unit ball in $H_2(X, \partial X) \otimes \mathbb{R}$ is at most $(2m+1)^b$.

Proof

Then the number of integral points in the unit ball for $H^2(X, \partial X) \otimes \mathbb{R}$ is at most $(2m+1)^b$:

Let $e^1, \ldots, e^b$ be the basis for $H^2(X, \partial X) \otimes \mathbb{R}$ dual to $S_1, \ldots, S_b$.
Let $u = \alpha_1 e^1 + \cdots + \alpha_b e^b$ be integral and in the unit ball.
Each α_i is integral:
Since u is integral, its evaluation against any element of $H_2(X, \partial X; \mathbb{Z})$ is integral. In particular $\alpha_i = u([S_i])$ is integral.
Since u is in the unit ball and $[S_i]/\chi_-(S_i)$ has norm 1, we deduce that $|u([S_i]/\chi_-(S_i))| \leq 1$. In other words,

$$|\alpha_i| = |u([S_i])| \leq \chi_-(S_i) \leq m.$$

So there are $(2m+1)$ possibilities for each α_i.

Controlling the genus of surfaces

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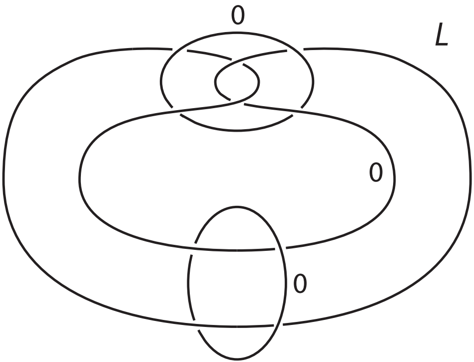

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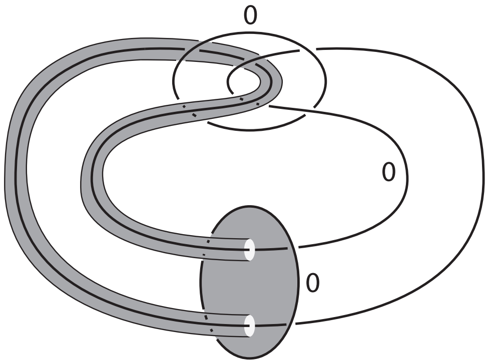

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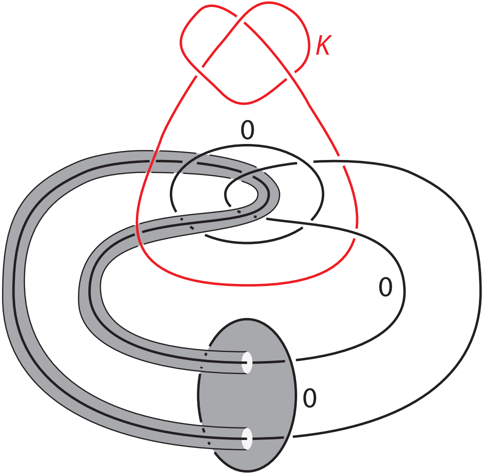

Controlling the genus of surfaces

- Seifert's algorithm can be used to produce a basis for $H_2(X, \partial X)$ with controlled χ_-.
- In fact we use a procedure involving cocycles and linking numbers in S^3.
- It is important here that we are dealing with a fixed 3-manifold M.
- The reason is that we use the linking matrix A of L.
- We need to find the inverse of a submatrix of A, which may end up being large.
- But for fixed M, A is fixed.

Other parts of the certificate

- We need to certify that $g(K) = g$.
- We do this by certifying the Thurston normal ball for $M \backslash\backslash N(K)$.
- We now know that it has at most polynomially many faces.
- We can certify the Thurston norm of the vertices and the barycentres of the faces using my certificate for Thurston norm.
- We show that we have found the entire boundary of the norm ball using the theory of pseudo-manifolds.
- On each face, we use Lenstra's algorithm for 'mixed integer programming'.
- We must also deal with spheres, discs, tori and annuli, using the theory of Tollefson and Wang.

Bounded b_1

We used that $b_1(M)$ is bounded at several points in the argument. Is this enough?

Question: Let b be a fixed natural number. Is the problem of determining the genus of a knot K in a compact orientable 3-manifold M with $b_1(M) \leq b$ in NP and co-NP?