39 Calculation of K-theory of crossed products via homotopy theory

initiated by discussions with J. Kranz

- now joint project with S. Nishikawa

39.1 First reduction to finite groups

Problem:

- G a group (discrete for this talk)

- A a C^*-algebra with G-action

- can form $A \rtimes_r G$

Calculate $K(A \rtimes_r G)$ from $K(A)$ with G-action.

Prototypical example:

- $G = \mathbb{Z}$

- $\alpha : K_*(A) \to K_*(A)$ action of the generator on K-theory groups

- Pimsner-Voiculescu sequence:

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha} & K_0(A) & \longrightarrow & K_0(A \rtimes \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow{1-\alpha} & K_1(A) \end{array}$$

- is in general not sufficient for calculation: extension problem

knowing G-action on spectrum suffices:

- have $K(A) \in \mathbf{Fun}(B\mathbb{Z}, \mathbf{Mod}(KU))$

– Baum-Connes for $\mathbb{Z}$ holds:

– $K(A \rtimes \mathbb{Z}) \simeq \mathtt{colim}_{BZ} K(A)$ (homotopy orbits)

- this is a homotopic theoretic description of spectrum $K(A \rtimes \mathbb{Z})$

– spectral sequence for groups leads again to Pimsner-Voiculescu

must ask question more precisely:

- in the present talk ask for calculation of the spectrum $K(A \rtimes_r G)$

- calculation means: description (formula) in terms of homotopy theory

– so $K(A \rtimes_r G) \simeq \mathtt{colim}_{BG} K(A)$ is elegible answer (not always true!)

– from such a formula to $K_*(A \rtimes_r G)$: spectral sequences and other methods

— carry this out this is not easy in general

must also be more precise about the data we want to use:

with increasing complexity

- $K_*(A) \in \mathbf{Fun}(BG, \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}})$

- $K(A) \in \mathbf{Fun}(BG, \mathbf{Mod}(KU))$ - here action by functoriality

- $\underline{K(A)} \in \mathbf{Fun}(G\mathbf{Orb}, \mathbf{Mod}(KU))$ - (see below for construction)

$\mathrm{kk}^G(A) \in \mathrm{KK}^G$

- everything depends on $\mathrm{kk}^G(A)$ - contains full information

$$
\begin{aligned}
\mathrm{KK}^G \quad &\stackrel{Davis-Lueck}{\to} \quad \mathbf{Fun}(G\mathbf{Orb}, \mathbf{Mod}(KU)) \\
&\stackrel{\mathrm{Res}_{\mathrm{Fin}}}{\to} \quad \mathbf{Fun}(G_{\mathrm{Fin}}\mathbf{Orb}, \mathbf{Mod}(KU)) \\
&\stackrel{\mathrm{Res}^{\mathrm{Fin}}}{\to} \quad \mathbf{Fun}(BG, \mathbf{Mod}(KU)) \\
&\stackrel{\pi_0}{\to} \quad \mathbf{Fun}(BG, \mathbf{Ab}^{\mathbb{Z}\mathrm{gr}})
\end{aligned}
$$

Davis-Lück construction (corrections by M. Joachim):

- $\mathbb{C}[-] : G\mathbf{Orb} \to GC^*\mathbf{Cat}^{\mathrm{nu}}$

- $S \in G\mathbf{Orb} \rightsquigarrow \mathbb{C}[S]$ - linearization in $GC^*\mathbf{Cat}^{\mathrm{nu}} \rightsquigarrow \mathrm{kk}^G(\mathbb{C}[S]) \in \mathrm{KK}^G$

- define $\underline{K(A)}$: $G\mathbf{Orb} \ni S \mapsto \mathrm{KK}^G(\mathbb{C}, A \otimes \mathbb{C}[S]) \in \mathbf{Mod}(KU)$

– values: $\underline{K(A)}(G/H) \simeq K(A \rtimes_r H)$

– $\underline{K(A)}$ contains full information: $\underline{K(A)}(G/G) \simeq K(A \rtimes_r G)$

Baum-Connes proposes:

- $\underline{K(A)}$ is left Kan-extension of $\underline{K(A)}_{|G_{\mathrm{Fin}}\mathbf{Orb}}$

– $\mathtt{colim}_{G_{\mathrm{Fin}}\mathbf{Orb}} \underline{K(A)} \simeq K(A \rtimes_r G)$

- if we accept Baum-Connes: must still calculate $\underline{K(A)}_{|G_{\mathrm{Fin}}\mathbf{Orb}}$

– in particular crossed products for finite groups

– next section - further reduction to finite cyclic subgroups

– $\underline{K(A)}$ is left Kan-extension of $\underline{K(A)}_{|G_{\mathcal{C}\mathrm{yc}\cap\mathrm{Fin}}\mathbf{Orb}}$

39.2 Second reduction to cyclic groups

$R(G) := \mathrm{KK}^G(\mathbb{C}, \mathbb{C}) \in \mathbf{CAlg}(\mathbf{Mod}(KU))$ representation ring of G

- KK^G is enriched in $\mathbf{Mod}(R(G))$

– $\underline{K(A)} \in \mathbf{Fun}(G\mathbf{Orb}, \mathbf{Mod}(R(G))$

– get in particular $K(A \rtimes H) \in \mathbf{Mod}(R(H))$ for all $H \subseteq G$

from now on: G finite

- $\pi_0 R(G)$ - classical representation ring

- $V := L^2(G) \ominus \mathbb{C}$ - ortho-complement of constant functions with G-action

- define $\xi := \Lambda_{-1}(V) := \sum_{i=0}^{|G|-1} (-1)^i \Lambda^i V$ in $\pi_0(G)$

$G_{\mathcal{P}\mathrm{rp}}\mathbf{Orb}$ - family of proper subgroups

- version of Atiyah-Segal

Theorem 39.1. *The isotropy separation fibre sequence*

$$\underset{G_{\mathcal{P}\mathrm{rp}}\mathbf{Orb}}{\mathtt{colim}}\, \underline{K(A)} \to K(A \rtimes G) \to \mathrm{Cof}^G_{\mathcal{P}\mathrm{rp}}(A)$$

is equivalent to the localization sequence

$$S_\xi K(A \rtimes G) \to K(A \rtimes G) \to K(A \rtimes G)[\xi^{-1}]$$

Lemma 39.2. *If G is not cyclic, then $\xi = 0$.*

- hence $\mathrm{Cof}^G_{\mathcal{P}\mathrm{rp}}(A) \simeq 0$ if G not cyclic

G again be general discrete:

Corollary 39.3. *For discrete group G the functor $\underline{K(A)}_{|G_{\mathrm{Fin}}\mathbf{Orb}}$ is left Kan-extension of $\underline{K(A)}_{|G_{\mathcal{C}\mathrm{yc}\cap\mathrm{Fin}}\mathbf{Orb}}$.*

with Baum-Connes: $\mathtt{colim}_{G_{\mathcal{C}\mathrm{yc}\cap\mathrm{Fin}}\mathbf{Orb}}\, \underline{K(A)} \simeq K(A \rtimes G)$.

- it suffices to calculate crossed products for cyclic groups

39.3 Reduction to families of subgroups

G finite

- for family of subgroups $\mathcal{F}$ consider isotropy separation fibre sequence:

$$\underset{G_{\mathcal{F}}\mathbf{Orb}}{\mathtt{colim}}\, \underline{K(A)} \to K(A \rtimes G) \to \mathrm{Cof}^G_{\mathcal{F}}(A)$$

have induction functors: $\mathtt{Ind}_H^G : \mathrm{KK}^H \to \mathrm{KK}^G$

- $I(\mathcal{F}) \subseteq \mathrm{KK}^G$ - localizing category generated by images of $\mathtt{Ind}_H^G$ for all members of $\mathcal{F}$

Proposition 39.4. *If $A \in I(\mathcal{F})$, then* $\mathrm{Cof}^G_{\mathcal{F}}(A) \simeq 0$.

Corollary 39.5. *If $A \in I(\mathcal{F})$, then $\underline{K(A)}$ is left Kan-extension of $\underline{K(A)}_{|G_{\mathcal{F}\cap\mathcal{C}\mathrm{yc}}\mathbf{Orb}}$ and* $\mathtt{colim}_{G_{\mathcal{F}\cap\mathcal{C}\mathrm{yc}}\mathbf{Orb}}\,\underline{K(A)} = K(A \rtimes G)$.

discuss case $\mathcal{F} = \{e\}$

- $G_{\{e\}}\mathbf{Orb} \simeq BG$

- $\underline{K(A)}_{|G_{\{e\}}} \simeq K(A)$ in $\mathbf{Fun}(BG, \mathbf{Mod}(KU))$

Corollary 39.6. *A in $I(\{e\})$, then* $\mathtt{colim}_{BG}\, K(A) \simeq K(A \rtimes G)$

- recovers Green-Julg $K(\mathtt{Ind}^G(B) \rtimes G) \simeq K(B)$

– $K(\mathtt{Ind}^G(B)) \simeq \bigoplus_G K(B)$

– $\mathtt{colim}_{BG} \bigoplus_G M \simeq M$

Proposition 39.7. *$A \in I(\{e\})$ and $|G|$ acts invertibly on A iff ρ acts invertibly on A.*

- in this case even $\mathtt{colim}_{BG}\, K_*(A) \simeq K_*(A \rtimes G)$

switch from KK^G to E^G (better behaved colimits)

- E^G is (probably) not compactly generated

– this makes the following interesting

- consider any set of objects $L \subseteq \mathrm{E}^G$

$$\bar{L} := \bigcap_{F \in \mathbf{Fun}^{\mathrm{colim}}(\mathrm{E}, \mathbf{Sp}), F(L)=0} \mathtt{ker}(F)$$

is (contains) closure of $\langle L \rangle$ under phantom retracts

– $K : \mathrm{E} \to \mathbf{Sp}$ is colimit preserving

$I(\{e\}) \subseteq \overline{\mathtt{Ind}^G(\mathrm{KK})}$ - probably proper

Corollary 39.8. *If A is in $\overline{\mathtt{Ind}^G(\mathrm{KK})}$, then $\underline{K(A)}$ is left Kan-extension of $K(A)$ and $\mathtt{colim}_{BG} K(A) \simeq K(A \rtimes G)$.*

Proposition 39.9. *If the G-action on A has the Rokhlin property, then $\mathrm{e}^G(A) \in \overline{\mathtt{Ind}^G(\mathrm{KK})}$.*

($\exists((p_{g,n})_{g\in G})_{n\in\mathbb{N}}$ with approximate properties: projections, decomposition of 1 in $M(A)$, equivariant, central)

more closure properties of $\overline{L}$

Proposition 39.10. *If A is approximately unitarily equivalent to B with $\mathrm{e}(B) \in \overline{L}$, then $\mathrm{e}(A) \in \overline{L}$.*

- ($f : A \to B$, $h : B \to A$, $fh \sim_{au} \mathtt{id}_B$, $hf \sim_{au} \mathtt{id}_A$)

- ($\sim_{au}$: $f \sim_{au} f' : A \to B$: $\exists(u_n)_n$ unitaries in B, $u_n f u_n^* \to f'$)

Proposition 39.11. *If $0 \to A \to B \to C \to 0$ is a weakly quasi-diagonal extension and $\mathrm{e}(B) \in \overline{L}$, then $\mathrm{e}(A), \mathrm{e}(C) \in \overline{L}$.*

($\exists(p_n)_{n\in\mathbb{N}}$ of invariant projections, $p_n B \subseteq A$, approximately central in B: $[p_n, b] \to 0$, approximate unit for a: $p_n a \to a$)

39.4 p-order cyclic groups

$G = C_p$

- has only e as proper subgroup

- fibre sequence

$$\mathop{\mathtt{colim}}_{BC_p} K(A) \to K(A \rtimes C_p) \to K(A \rtimes C_p)[\xi^{-1}]$$

Lemma 39.12. $\xi^2 = p\xi$

hence completing at p kills third term

Corollary 39.13.

$$(\mathop{\mathtt{colim}}_{BC_p} K(A))^{\wedge}_p \simeq K(A \rtimes C_p)^{\wedge}_p$$

suffices to calculate p-torsion in $K_*(A \rtimes C_p)$

- $K(A) \simeq 0$ (non-equivariantly) implies $\mathtt{colim}_{BC_p} K(A) \simeq 0$ and hence $K(A \rtimes C_p)$ is uniquely p-divisible (actually ξ-divisible)

- Iszumi: there are many examples of such A

Corollary 39.14. *If $K_*(A)$ is finite p-torsion, then no completion necessary:*

$$\underset{BC_p}{\mathtt{colim}}\, K(A) \simeq K(A \rtimes C_p)$$

calculation of homotopy is still complicated

example:

$A = \mathcal{O}_{p^n+1}^{\otimes C_p}$ with cyclic permutation of tensor factors

- $K(A) \simeq (KU/p^n)^{n_0} \oplus (\Sigma KU/p^n)^{n_1}$, n_0, n_1 explicitly known

– e.g for $p = 2$, $n = 1$: $n_0 = n_1 = 1$

Corollary 39.14 applies

$$\underset{BC_p}{\mathtt{colim}}\, K(\mathcal{O}_{p^n+1}^{\otimes C_p}) \simeq K(\mathcal{O}_{p^n+1}^{\otimes C_p} \rtimes C_p)$$

homotopy groups only known for $p = 2, 3$ and all n: Izumi, Nishikawa

$$K_*(\mathcal{O}_{2^n+1}^{\otimes C_2} \rtimes C_2) \cong \begin{cases} \mathbb{Z}/2^{n+1} \oplus \mathbb{Z}/2^{n-1} & * = 0 \\ 0 & * = 1 \end{cases}$$

(joint with Nishikawa:)

$$K_*(\mathcal{O}_{3^n+1}^{\otimes C_3} \rtimes C_3) \cong \begin{cases} (\mathbb{Z}/3^{n+1})^2 \oplus (\mathbb{Z}/3^{n-1})^2 & * = 0 \\ 0 & * = 1 \end{cases}$$

Method is not simply evaluate formula above!

- uses fine structure of tensor powers - filtrations