Instanton Floer homology and the depth of taut foliations.

May 26 th.

Defn (Balanced sutured manifold).

A pair (M, γ) is called a balanced sutured mfd.

- M is cpt, oriented 3-mfd. $\partial M \neq \emptyset$.
- $\gamma \subseteq \partial M$ oriented closed 1-submfd.

$A(\gamma) = [-1,1] \times \gamma \subseteq \partial M$. $R(\gamma) = \partial M \setminus A(\gamma)$.

(1). ∂M, $R(\gamma)$ have no closed component.

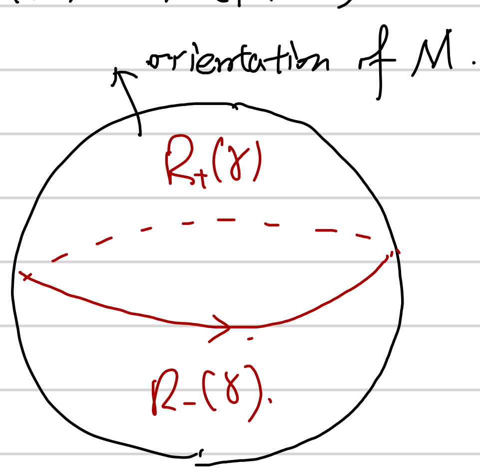
(2). The orientation of γ induces an orientation on $R(\gamma)$.

$R_+(\gamma)$ = part of $R(\gamma)$ two orientations (from γ, M) coincide.

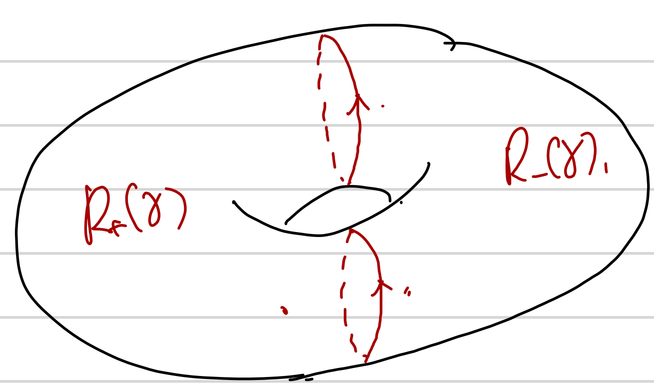
$R_-(\gamma) = R(\gamma) \setminus R_+(\gamma)$.

(3). $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$.

E.g. -



B^3.



$= S^1 \times D^2$.

Defn (Thurston norm). A properly embedded surface $(S, \partial S) \subseteq (M, \partial M)$

$$x(S) = \begin{cases} \max\{-\chi(S), 0\}. & S \text{ is connected.} \\ x(S_1) + \cdots + x(S_n) & S = S_1 \cup \cdots \cup S_n \text{ connected component.} \end{cases}$$

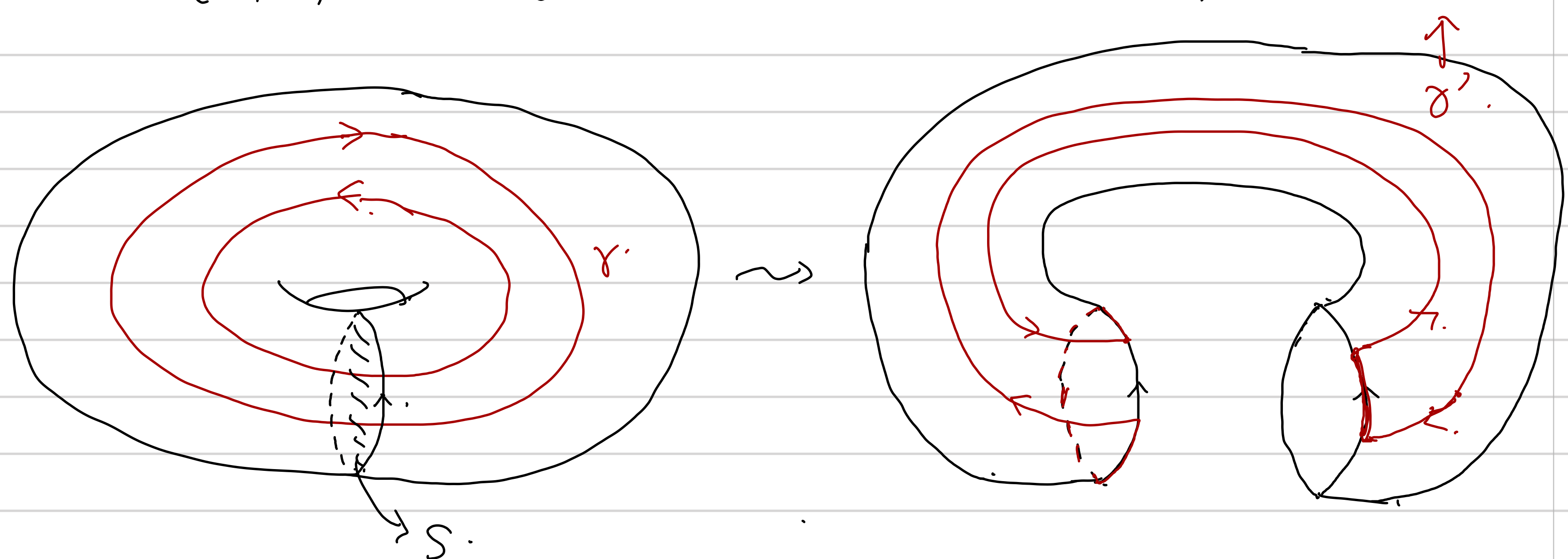
Defn. A balanced sutured mfd is called taut if

- M is irreducible.
- $R_\pm(\gamma)$ are both incompressible.
- $R_\pm(\gamma)$ minimizes the Thurston norms of their homolog classes.

Sutured manifold decompositions.

$(S, \partial S) \subseteq (M, \partial M)$. S is oriented

$(M, \gamma) \overset{S}{\rightsquigarrow} (M', \gamma')$. $\quad M' = M \setminus N(S)$.



Solid torus.

Thm (Gabai, 1983).

Suppose (M, γ) taut balanced sutured manifold.

Then there is a finite sequence of decompositions:

$(M, \gamma) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1) \rightsquigarrow \cdots \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n)$.

So that:

(1) Each (M_i, γ_i) is taut.

(2). $(M_n, \gamma_n) = ([-1,1] \times F, \{0\} \times \partial F)$. (product sutured manifold.

F: oriented cpt surface, with no closed component.

Defn. The sequence.

$$(M,\gamma) \overset{S_1}{\rightsquigarrow} (M_1,\gamma_1) \rightsquigarrow \cdots \overset{S_n}{\rightsquigarrow} (M_n,\gamma_n).$$

is called a sutured mfd hierarchy.

$d(M,\gamma)$ = minimal possible n.

$d(M,\gamma)$ = depth of the balanced sutured mfd.

Q: How finite can $d(M,\gamma)$ be?

A: Juhász $SFH(M,\gamma)$.

Thm (Juhász, 2010). If $H_2(M;\mathbb{Z})=0$ (M,γ) taut.

$$rk_{\mathbb{Z}}\, SFH(M,\gamma) < 2^{k+1}.$$

Then $d(M,\gamma) \le 2k$.

Thm (Ghosh, L, 2019). If $H_2(M;\mathbb{Z})=0$. (M,γ) taut.

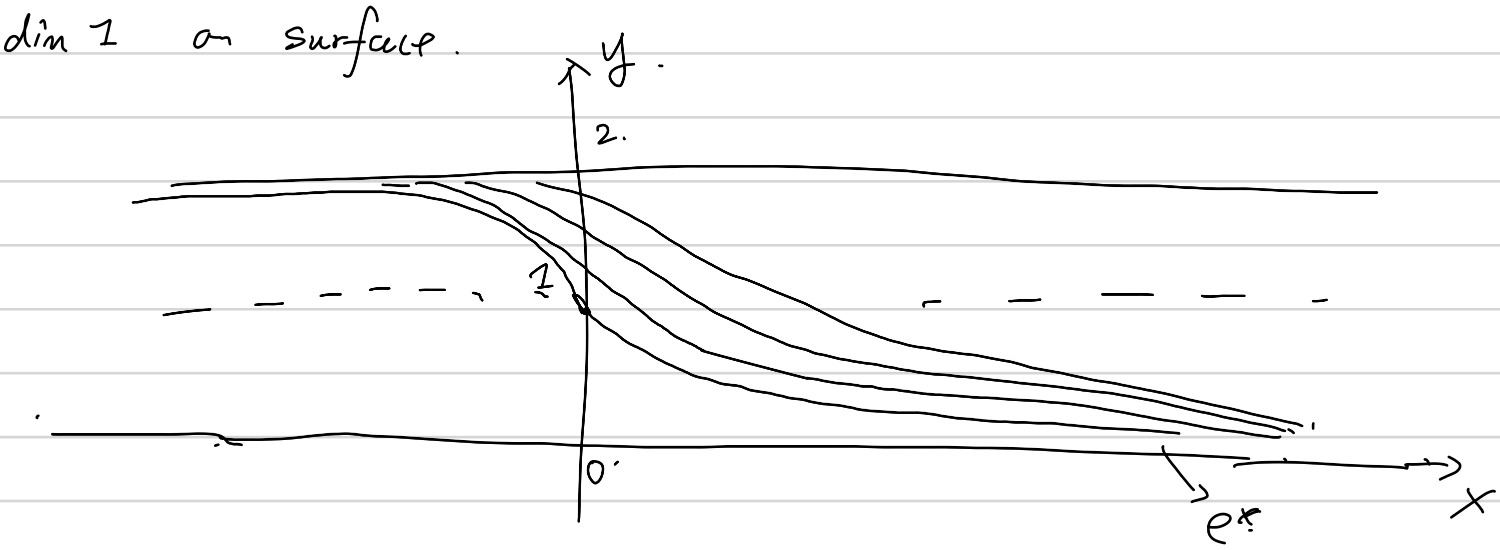
$$\dim_{\mathbb{C}} SHI(M,\gamma) < 2^{k+1}.$$

Then $d(M,\gamma) \le 2k$.

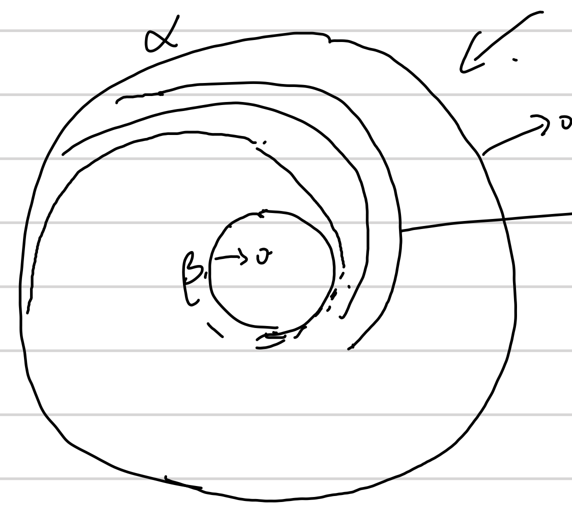
Foliations: (co-dim 1 on 3-mfd).

Locally: x-y, $-z$ plane. leaves looks like horizontal planes. parametrized by z-axis.

E.g. co-dim 1 on surface.



SQuotient to annulus:



$\bar{L}$ contains $\alpha \cup \beta \cup L$.

Defn. A foliation $\mathcal{F}$ on (M, γ) is called taut if

- $\mathcal{F}$ is transverse to $A(\gamma) = [-1,1] \times \gamma \subseteq \partial M$.
- $\mathcal{F}$ is tangent to $R(\gamma)$.
- $\mathcal{F}|_{A(\gamma)}$ has no Reeb components.
- Each leaf of $\mathcal{F}$ intersects a transverse curve or properly embedded arcs.

E.g. Say $(M, \gamma) = ([-1,1] \times F, \{0\} \times \partial F)$. Then there is a product foliation.

leaves are $\{t\} \times F$ $t \in [-1,1]$.

Defn - Depth of the foliation: defined inductively.

- Compact leaves will have depth 0.
- If L is a leaf. $\bar{L}$ contains only leaves of depth 0.

Then say depth of L is 1.

Depth of a foliation = maximal depth of all its leaves.

Thm (Gabai). If (M, γ) is taut then it admits a taut foliation of finite depth.

Q: How finite it could be?

Conj. (Juhász). $H_2(M)=0$. $rk_{\mathbb{Z}} SFH(M,\gamma) < 2^{k+1}$ (M,γ) taut.

Then (M,γ) admits a taut foliation of depth $\leq 2k$.

Thm (L). $H_2(M)=0$. $\dim_{\mathbb{C}} SHI(M,\gamma) < 2^{k+1}$. (M,γ) taut.

Then (M,γ) admits a taut foliation of depth 2^{k+6}.

also works for SFH.

Cor. $K \subseteq S^3$ is a knot. Consider irreducible $SU(2)$ reps.

$$\rho: \pi_1(S^3 \setminus K) \to SU(2).$$

$$\rho(\text{meridian}) = \vec{i} \in SU(2).$$

Suppose those homomorphisms are all non-degenerate and

$n(K)$ = number of conjugacy classes of such homomorphism

$n(K) < 2^{k+1}$.

Then $S^3(K)$ admits a taut foliation of depth $\leq 2^{k+6}$, transverse to the boundary.

Conj (Kronheimer, Mrowka, 2010). depth $\leq 2k$.

Sutured instanton Floer homology.

(M,γ), balanced suture mfd. $\rightsquigarrow$ $SHI(M,\gamma)$. finite dim complex vector space.

Thm (Baldwin–Sivek). Well defined up to multiplication by an element in $\mathbb{C}^*$.

Thm (Ghosh, L, 2019). $H_2(M)=0$ (M,γ) taut

$$\dim_{\mathbb{C}} SHI(M,\gamma) < 2^{k+1}.$$

Then $d(M,\gamma) \le 2k$.

Idea: Do decomposition in k stage, each stage: 2 decompositions

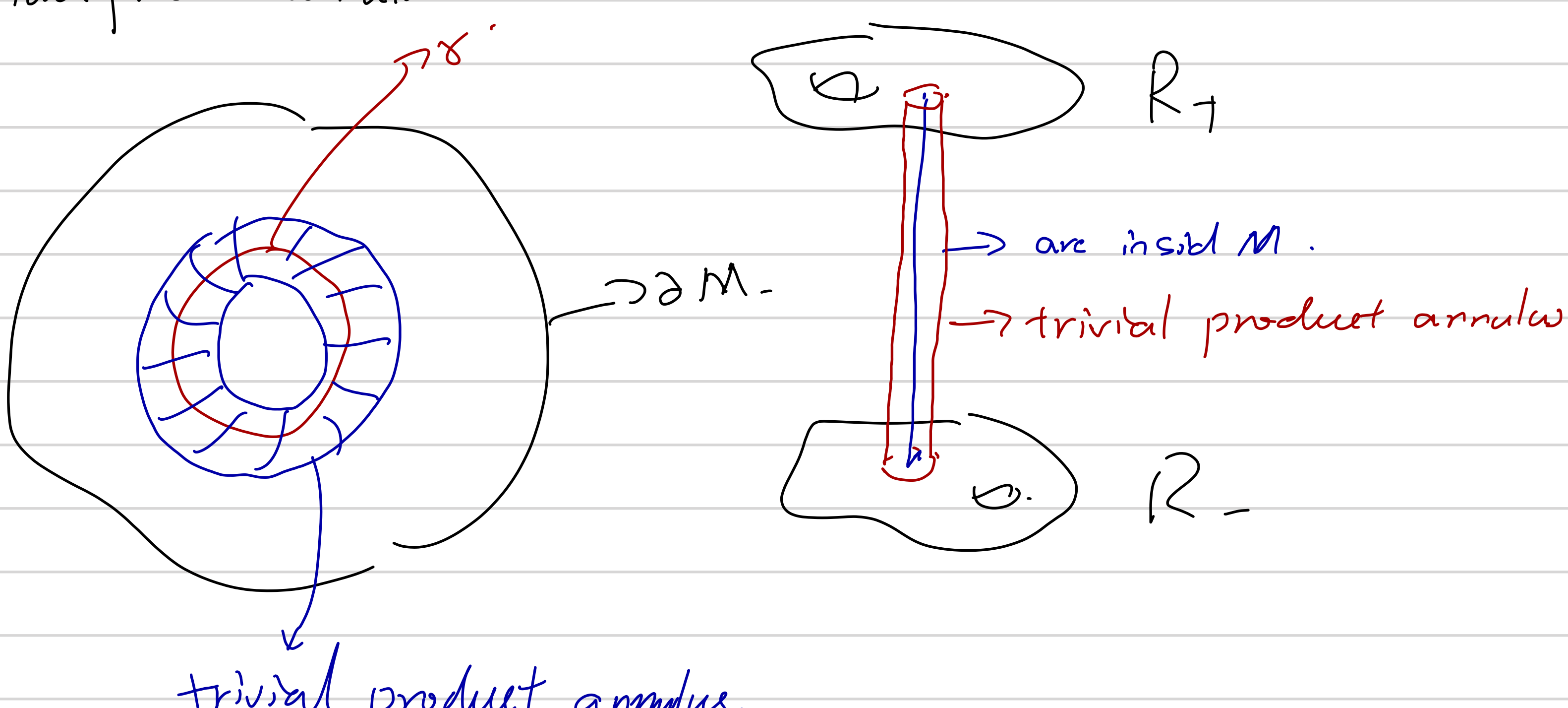
$\dim_{\mathbb{C}} SHI(M,\gamma)$ reduced by half.

In each step:

Defn. product annulus: $(A,\partial A) \subseteq (M, R(\gamma))$,

$$\partial A \cap R_+(\gamma) = \phi. \quad \partial A \cap R_-(\gamma) = \phi.$$

Trivial product annulus:



In each stage: $\dim^{SHI}(M_1,\gamma_1) \le \dim^{SHI}(M,\gamma)$, $\quad \dim^{SHI}(M_2,\gamma_2) \le \frac{1}{2}\dim SHI(M_1,\gamma_1)$

$(M,\gamma) \rightsquigarrow (M_1,\gamma_1) \rightsquigarrow (M_2,\gamma_2)$,

maximal collection of disjoint non-trivial product annulus.

free of non-trivial product annulus.

∇. well-groomed surface.

Pick any $\alpha \in H_2(M,\partial M)$

Pick a surface $[S] = \pm\alpha$

$(M_1,\gamma_1) \overset{S}{\rightsquigarrow} (M_2,\gamma_2)$.

Prop (Gabai). $(M,\gamma) \overset{S}{\rightsquigarrow} (M',\gamma')$ (M,γ), (M',γ') taut.

S is well-groomed. $\mathcal{F}'$ is a taut foliation on (M',γ') of depth $\underset{\circ}{k}$.

Then (M,γ) admist a taut foliation of depth $\underset{\circ}{k+1}$.

Lemma: The decomposition along each product annulus can be divides into a sequence of well-groomed decompositions. with total number $\leq C$.

$\uparrow$ constant.

Remaining: to bounds the number of product annulus.

Observation: # product annulus $\leq$ # component of the suture γ_1

Lemma (Ghosh, L, 2019).

component of suture on $\gamma_1 \leq C' \cdot \dim SHI(M_1,\gamma_1)$.