38 Introduction to *KK*

in this talk: all C^* -algebras are separable

- $C^* \mathbf{Alg}^{\mathrm{nu}}$ - category of C^* -algebras

Kasparov groups (Pierre's talk)

- A, B in C^*Alg^{nu} :
- $KK_0(A, B)$ Kasparov's KK-theory group
- homotopy classes of Kasparov modules $[E, \phi, F]$

features

- $f: A \to B$ induces $\mathrm{kk}_0(f) := [B, f, 0] \in \mathrm{KK}_0(A, B)$
- Kasparov product: \circ : KK₀(B, C) \otimes KK₀(A, B) \rightarrow KK₀(A, C)
- is associative, compatible with composition of morphisms

end code this structure as $\mathbbm{Z}\text{-endriched}$ category KK_0

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- objects: C^*-algebras
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- morphisms: $\operatorname{Hom}_{\operatorname{KK}_0}(A, B) := \operatorname{KK}_0(A, B)$
- composition: \circ

express naturality properties:

Proposition 38.1. We have a functor $kk_0 : C^*Alg^{nu} \to KK_0$.

further properties:

- homotopy invariance: $A \mapsto C([0,1]) \otimes A$ goes to iso
- K-stability: $A \cong \mathbb{C} \otimes A \xrightarrow{p \otimes id_A} K \otimes A$ goes to iso

- split exact: for split exact $0 \longrightarrow A \longrightarrow B \xrightarrow{\longleftarrow} C \longrightarrow 0$ the sequence $0 \longrightarrow kk_0(A) \longrightarrow kk_0(B) \xrightarrow{\longrightarrow} kk_0(C) \longrightarrow 0$ presents $kk_0(B) \cong kk_0(A) \oplus kk_0(C)$

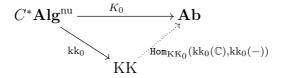
- kk_0 sends sums to sums

Theorem 38.2 (Higson). KK₀ is an additive category and $kk_0 : C^*Alg_s^{nu} \to KK_0$ is the initial homotopy invariant, K-stable and split exact functor to an additive category.

$$\mathrm{kk}_{0}^{*}:\mathbf{Fun}^{\mathrm{add}}(\mathrm{KK}_{0},\mathbf{A})\stackrel{\simeq}{
ightarrow}\mathbf{Fun}^{h,K,split}(C^{*}\mathbf{Alg}^{\mathrm{nu}},\mathbf{A})$$

example: $K_0: C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Ab}$ is homotopy invariant K-stable and split exact

- has unique factorization:



- for $F: C^*Alg^{nu} \to Ab$ homotopy invariant, split-exact and K-stable:

– by Yoneda Lemma Nat $(K_0, F) \cong F(\mathbb{C})$

encode external products

Proposition 38.3. KK_0 has a unique bi-additive symmetric monoial structure such that $kk_0 : C^*Alg^{nu} \to KK_0$ is symmetric monoidal

(two statements, one for for \otimes_{\min} and one for \otimes_{\max})

define $\Omega := \mathrm{kk}_0(C_0(\mathbb{R})) \otimes - : \mathrm{KK}_0 \to \mathrm{KK}_0$

- additional structure: boundary maps
- consider semiexact $0 \longrightarrow A \longrightarrow B \xrightarrow{\cong} C \longrightarrow 0$
- induces $\partial : \Omega \operatorname{kk}_0(C) \to \operatorname{kk}_0(A)$ (explicit construction on cycle level)

Proposition 38.4.

$$\Omega \mathrm{kk}_0(C) \xrightarrow{\partial} \mathrm{kk}_0(A) \to \mathrm{kk}_0(B) \to \mathrm{kk}_0(C)$$
(38.1)

 $is \ exact.$

Toplitz sequence $0 \longrightarrow K \longrightarrow \mathcal{T}_0 \xrightarrow{\Bbbk} \mathcal{C}_0(\mathbb{R}) \longrightarrow 0$ and $\mathrm{kk}_0(\mathcal{T}_0) \cong 0$ (Cuntz) gives Bott periodicity:

$$\Omega^2 \cong \Omega \mathrm{kk}_0(C_0(\mathbb{R})) \otimes - \stackrel{\partial \otimes -}{\cong} \mathrm{kk}_0(K) \otimes - \cong \mathrm{id}$$

- use this to define Ω^i for i < 0

Proposition 38.5 (Meyer-Nest). KK_0 has structure of a triangulated category with shift Ω and triangles (38.1).

Auestions:

1.

- as above: *KK*-groups are defined using cycles by relations

- additional structures (composition, product, boundaries) must be constructed on cycle level

- very technical arguments for well-defineness and properties

Can one show by abstract nonsense that there exists a functor with the universal property of $kk_0 : C^* Alg^{nu} \to KK_0$ from Proposition 38.3?

- It would be unique up to unique isomorphism.

– Get all the structures for free and in a simple manner (except boundaries).

- Main remaining problem in this approach: Must relate the resulting Hom-groups in this category with Kasparov modules.

2.

- usually triangulated categories are homotopy categories of stable ∞ -categories

Can one construct a stable ∞ -category KK with homotopy category KK₀ and a refinement $kk : C^*Alg^{nu} \to KK$ of kk_0 .

- Existence of boundary operators for semiexact sequences becomes property.

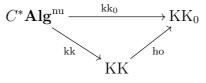
classical Answer to 1. is Higsons construction

Quick solution for 2. by Land-Nikolaus:

- W - kk₀ - equivalences in $C^* \mathbf{Alg}^{\mathrm{nu}}$

Define: kk : $C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}[W^{-1}] =:$ KK by Dwyer-Kan localization

Proposition 38.6 (Land-Nikolaus). KK is a stable ∞ -category and



commutes. All works with symmetric monoidal structures \otimes_{max} and \otimes_{min} .

proof uses fibration category (Uuye):

- weak equivalences: kk₀-equivalences

- fibrations: semi-split surjections

combine this with classical constructions:

Corollary 38.7. kk is homotopy invariant, K-stable and sends semi-exact sequences to fibre sequences.

Lecture course question: How can one tell students what KK is without explaining KK_0 ?

notions of exactness for functor F:

 $-? \in \{$ Schochet fibration, split, semisplit, surjective $\}$

- F is ?-exact: it sends cartesian square with property ? to cartesian square

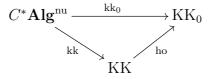
$$\begin{array}{cccc} C \longrightarrow A & \mapsto & F(C) \longrightarrow F(A) \\ \downarrow & \downarrow ? & & \downarrow & \downarrow \\ D \longrightarrow B & & F(D) \longrightarrow F(B) \end{array}$$

- $f: A \to B$ is Schochet fibration of $\operatorname{Hom}_{C^*Alg^{nu}}(D, A) \to \operatorname{Hom}_{C^*Alg^{nu}}(D, B)$ is Serre fibration

Theorem 38.8. There is an initial Schochet- and split exact, homotopy invariant and K-stable functor $kk : C^*Alg^{nu} \to KK$ to a left-exact and additive ∞ -category. All works with symmetric monoidal structures \otimes_{max} and \otimes_{min} .

$$kk^*$$
: Fun^{lex}(KK, A) $\xrightarrow{\simeq}$ Fun^{h,K,split+sch}(C*Alg^{nu}, A)

- commutes by comparison of universal properties



- 1. $L_h: C^* \mathbf{Alg}^{\mathrm{nu}} \to C^* \mathbf{Alg}_h^{\mathrm{nu}}$ Dwyer-Kan at homotopy equivalences
 - get left exact pointed ∞ -category
 - $\operatorname{Map}_{C^*\operatorname{Alg}_{k}^{\operatorname{nu}}}(A, B) \simeq \ell \operatorname{Hom}(A, B)$ (Anima associated to topological Hom-space)
 - L_h is Schochet exact
- 2. $\mathbb{C} \to K$ is tensor idempotent in $C^* \mathbf{Alg}_h^{\mathrm{nu}}$
 - left Bousfield localization $L_K := K \otimes : C^* \mathbf{Alg}_h^{\mathrm{nu}} \to L_K C^* \mathbf{Alg}_h^{\mathrm{nu}}$
 - inverts $A \to K \otimes A$
 - get left semi-additive left-exact ∞ -category
 - $L_K \circ L_h$ is Schochet-exact and K-stable
 - $\operatorname{Map}_{L_K C^* \operatorname{Alg}_h^{\operatorname{nu}}}(A, B) \simeq \ell \operatorname{Hom}(A, K \otimes B)$
- 3. force split exactness:
 - in $L_K C^* \mathbf{Alg}_h^{\mathrm{nu}}$: $A * A \simeq A \oplus A$

$$- 0 \longrightarrow qA \longrightarrow A * A * A \xrightarrow{fold} A \longrightarrow 0$$

 $0 \to qA \to A * A \xrightarrow{\checkmark} A \to 0, \ qA \to A$

- $qA \to A$

- any split exact functor should invert ι_A
- Dwyer-Kan localize at $W := \{\iota_A | A \in C^* \mathbf{Alg}^{\mathrm{nu}}\}$

 $-L_q: L_K C^* \mathbf{Alg}_h^{\mathrm{nu}} \to L_K C^* \mathbf{Alg}_h^{\mathrm{nu}}[W^{-1}] =: \mathrm{KK}$

- get additive left-exact ∞ -category (easy)
- $-\mathrm{kk} := L_{q,K,h} : C^* \mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ is homotopy invariant, K-stable and Schochet- and split exact and aninital with this property.

- calculus of fractions: $\operatorname{Map}_{\operatorname{KK}}(A, B) \simeq \operatorname{colim}_{n \in \mathbb{N}} \ell \operatorname{Hom}(q^n A, B \otimes K)$

This is a complete description of kk : $C^*Alg^{nu} \to KK$ independent of classical cycle by relations picture!

deep theorem of Cuntz: $q^2 A \to q A$ is equivalence in $L_K C^* \mathbf{Alg}_h^{\mathrm{nu}}$

– implies: L_q is right Bousfield localization

 $-\operatorname{Map}_{\operatorname{KK}}(A,B) \cong \ell \operatorname{Hom}(qA,B \otimes K)$

- this is one of the classical formulas for KK

stability, fibre sequences ?

interpretation of Cuntz-Skandalis:

Theorem 38.9 (Automatik semiexactess). For any left-exact additive ∞ -category **A** incl: $\operatorname{Fun}^{h,K,se+Sch}(C^*\operatorname{Alg}^{\operatorname{nu}}, \mathbf{A}) \to \operatorname{Fun}^{h,K,split+Sch}(C^*\operatorname{Alg}^{\operatorname{nu}}, \mathbf{A})$

is an equivalence.

apply to $kk : C^*Alg^{nu} \to KK$

- conclude: kk is semiexact
- note that $\Omega := \operatorname{kk}(C_0(\mathbb{R})) \otimes -$ is loop endofunctor of left-exact structure
- use Toeplitz extension as above: $\Omega^2 \simeq id_{KK}$

Corollary 38.10. KK is a stable ∞ -category with bi-leftexact symmetric monoidal structure.

- gives long exact sequences, Bott periodicity, triangulated structure on hoKK

 $se \implies Sch + split$ if target is stable:

Corollary 38.11. kk : $C^*Alg^{nu} \to KK$ is an initial homotopy invariant, K-stable and semi-exact functor to a stable ∞ -category. All works with symmetric monoidal structures \otimes_{\max} and \otimes_{\min} .

for any stable \mathbf{C}

$$\mathrm{kk}^*: \mathbf{Fun}^{ex}(\mathrm{KK}, \mathbf{C}) \xrightarrow{\simeq} \mathbf{Fun}^{h, K, se}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{C})$$

How fits *E*-theory in this picture?

for exact sequence $0 \to A \to B \to C \to 0$ get comparison map $kk(A) \to Fib(kk(B) \to kk(C))$

- equivalence if sequence is semisplit, but not in general

Dwyer-Kan W := all comparison maps

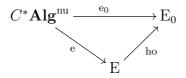
- get functor $c: \mathrm{KK} \to \mathrm{KK}[W^{-1}] =: \mathrm{E}$
- set $\mathbf{e} := c \circ \mathbf{kk} : C^* \mathbf{Alg}^{\mathrm{nu}} \mathbf{KK} \to \mathbf{E}$

Theorem 38.12. e : $C^*Alg^{nu} \to E$ is a Dwyer-Kan localization and the initial homotopy invariant, K-stable and exact functor to a stable ∞ -category. All works with symmetric monoidal structure \otimes_{max} .

$$e^* : \mathbf{Fun}^{ex}(E, \mathbf{C}) \xrightarrow{\simeq} \mathbf{Fun}^{h, K, ex}(C^* \mathbf{Alg}^{\mathrm{nu}}, \mathbf{C})$$

comparison with classical *E*-theory (Higson)

- commutes by comparison of universal properties



Get lax symmetric monoidal, homotopy invariant, K-stable and exact spectrum-valued K-theory functor for free:

$$K(-) := \operatorname{map}_{\mathcal{E}}(\mathbf{e}(\mathbb{C}), \mathbf{e}(-)) : C^* \mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Sp}$$

- for homotopy invariant, exact and K-stable $F:C^*\mathbf{Alg}^{\mathrm{nu}}\to \mathbf{Sp}$
- $-\operatorname{Nat}(K,F) \simeq \Omega^{\infty} F(\mathbb{C})$ (Yoneda Lemma)
- $-\pi_0 \operatorname{Nat}(K, K[1/p]) \cong \mathbb{Z}[1/p]$

application:

- for commutative algebras have $\Psi^p: K \to K[1/p]$ (Adams operation)
- does not extend to all $C^{\ast}\mbox{-algebras}$