38 Introduction to KK

in this talk: all C^*-algebras are separable

- $C^*\mathbf{Alg}^{\mathrm{nu}}$ - category of C^*-algebras

Kasparov groups (Pierre's talk)

- A, B in $C^*\mathbf{Alg}^{\mathrm{nu}}$:

– $\mathrm{KK}_0(A, B)$ - Kasparov's KK-theory group

– homotopy classes of Kasparov modules $[E, \phi, F]$

features

- $f : A \to B$ induces $\mathrm{kk}_0(f) := [B, f, 0] \in \mathrm{KK}_0(A, B)$

- Kasparov product: $\circ : \mathrm{KK}_0(B, C) \otimes \mathrm{KK}_0(A, B) \to \mathrm{KK}_0(A, C)$

– is associative, compatible with composition of morphisms

endcode this structure as $\mathbb{Z}$-endriched category KK_0

- objects: C^*-algebras

- morphisms: $\mathtt{Hom}_{\mathrm{KK}_0}(A, B) := \mathrm{KK}_0(A, B)$

- composition: $\circ$

express naturality properties:

Proposition 38.1. *We have a functor* $\mathrm{kk}_0 : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}_0$.

further properties:

- homotopy invariance: $A \mapsto C([0, 1]) \otimes A$ goes to iso

- K-stability: $A \cong \mathbb{C} \otimes A \stackrel{p\otimes \mathtt{id}_A}{\to} K \otimes A$ goes to iso

- split exact: for split exact $0 \longrightarrow A \longrightarrow B \overset{\curvearrowleft}{\longrightarrow} C \longrightarrow 0$ the sequence

$0 \longrightarrow kk_0(A) \longrightarrow \mathrm{kk}_0(B) \overset{\curvearrowleft}{\longrightarrow} \mathrm{kk}_0(C) \longrightarrow 0$ presents $\mathrm{kk}_0(B) \cong \mathrm{kk}_0(A) \oplus \mathrm{kk}_0(C)$

- kk_0 sends sums to sums

Theorem 38.2 (Higson)**.** KK_0 *is an additive category and* $\mathrm{kk}_0 : C^*\mathbf{Alg}^{\mathrm{nu}}_{\mathrm{s}} \to \mathrm{KK}_0$ *is the initial homotopy invariant, K-stable and split exact functor to an additive category.*

$$\mathrm{kk}_0^* : \mathbf{Fun}^{\mathrm{add}}(\mathrm{KK}_0, \mathbf{A}) \overset{\simeq}{\to} \mathbf{Fun}^{h,K,split}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{A})$$

example: $K_0 : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Ab}$ is homotopy invariant K-stable and split exact

- has unique factorization:

$$\begin{array}{ccc} C^*\mathbf{Alg}^{\mathrm{nu}} & \xrightarrow{K_0} & \mathbf{Ab} \\ & {\scriptstyle \mathrm{kk}_0}\searrow \quad \nearrow{\scriptstyle \mathtt{Hom}_{\mathrm{KK}_0}(\mathrm{kk}_0(\mathbb{C}),\mathrm{kk}_0(-))} & \\ & \mathrm{KK} & \end{array}$$

- for $F : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Ab}$ homotopy invariant, split-exact and K-stable:

– by Yoneda Lemma $\mathrm{Nat}(K_0, F) \cong F(\mathbb{C})$

encode external products

Proposition 38.3. KK_0 *has a unique bi-additive symmetric monoial structure such that* $\mathrm{kk}_0 : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}_0$ *is symmetric monoidal*

(two statements, one for for $\otimes_{\mathrm{min}}$ and one for $\otimes_{\mathrm{max}}$)

define $\Omega := \mathrm{kk}_0(C_0(\mathbb{R})) \otimes - : \mathrm{KK}_0 \to \mathrm{KK}_0$

- additional structure: boundary maps

- consider semiexact $0 \longrightarrow A \longrightarrow B \overset{\curvearrowleft}{\longrightarrow} C \longrightarrow 0$

- induces $\partial : \Omega\mathrm{kk}_0(C) \to \mathrm{kk}_0(A)$ (explicit construction on cycle level)

Proposition 38.4.

$$\Omega \mathrm{kk}_0(C) \stackrel{\partial}{\to} \mathrm{kk}_0(A) \to \mathrm{kk}_0(B) \to \mathrm{kk}_0(C) \tag{38.1}$$

is exact.

Toplitz sequence $0 \longrightarrow K \longrightarrow \mathcal{T}_0 \longrightarrow C_0(\mathbb{R}) \longrightarrow 0$ and $\mathrm{kk}_0(\mathcal{T}_0) \cong 0$ (Cuntz) gives Bott periodicity:

$$\Omega^2 \cong \Omega \mathrm{kk}_0(C_0(\mathbb{R})) \otimes - \stackrel{\partial\otimes -}{\cong} \mathrm{kk}_0(K) \otimes - \cong \mathtt{id}$$

- use this to define Ω^i for $i < 0$

Proposition 38.5 (Meyer-Nest)**.** KK_0 *has structure of a triangulated category with shift* Ω *and triangles* (38.1)*.*

Auestions:

1.

- as above: KK-groups are defined using cycles by relations

- additional structures (composition, product, boundaries) must be constructed on cycle level

– very technical arguments for well-defineness and properties

Can one show by abstract nonsense that there exists a functor with the universal property of $\mathrm{kk}_0 : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}_0$ from Proposition 38.3?

– It would be unique up to unique isomorphism.

– Get all the structures for free and in a simple manner (except boundaries).

– Main remaining problem in this approach: Must relate the resulting `Hom`-groups in this category with Kasparov modules.

2.

- usually triangulated categories are homotopy categories of stable ∞-categories

Can one construct a stable ∞-category KK with homotopy category KK_0 and a refinement $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ of kk_0.

- Existence of boundary operators for semiexact sequences becomes property.

classical Answer to 1. is Higsons construction

Quick solution for 2. by Land-Nikolaus:

- W - kk_0 - equivalences in $C^*\mathbf{Alg}^{\mathrm{nu}}$

Define: $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}[W^{-1}] =: \mathrm{KK}$ by Dwyer-Kan localization

Proposition 38.6 (Land-Nikolaus)**.** KK *is a stable ∞-category and*

$$\begin{array}{ccc} C^*\mathbf{Alg}^{\mathrm{nu}} & \xrightarrow{\mathrm{kk}_0} & \mathrm{KK}_0 \\ & {}_{\mathrm{kk}}\searrow \quad \nearrow_{\mathrm{ho}} & \\ & \mathrm{KK} & \end{array}$$

commutes. All works with symmetric monoidal structures $\otimes_{\max}$ *and* $\otimes_{\min}$.

proof uses fibration category (Uuye):

- weak equivalences: kk_0-equivalences

- fibrations: semi-split surjections

combine this with classical constructions:

Corollary 38.7. kk *is homotopy invariant, K-stable and sends semi-exact sequences to fibre sequences.*

Lecture course question: How can one tell students what KK is without explaining KK_0?

notions of exactness for functor F:

- $? \in \{$Schochet fibration, split, semisplit, surjective$\}$

- F is ?-exact: it sends cartesian square with property ? to cartesian square

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow ? \\ D & \longrightarrow & B \end{array} \quad \mapsto \quad \begin{array}{ccc} F(C) & \longrightarrow & F(A) \\ \downarrow & & \downarrow \\ F(D) & \longrightarrow & F(B) \end{array}$$

- $f : A \to B$ is Schochet fibration of $\mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(D,A) \to \mathtt{Hom}_{C^*\mathbf{Alg}^{\mathrm{nu}}}(D,B)$ is Serre fibration

Theorem 38.8. *There is an initial Schochet- and split exact, homotopy invariant and K-stable functor* $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ *to a left-exact and additive ∞-category. All works with symmetric monoidal structures $\otimes_{\max}$ and $\otimes_{\min}$.*

$$\mathrm{kk}^* : \mathbf{Fun}^{\mathrm{lex}}(\mathrm{KK}, \mathbf{A}) \xrightarrow{\simeq} \mathbf{Fun}^{h,K,split+sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{A})$$

- commutes by comparison of universal properties

$$\begin{array}{ccc} C^*\mathbf{Alg}^{\mathrm{nu}} & \xrightarrow{\mathrm{kk}_0} & \mathrm{KK}_0 \\ & {\scriptstyle \mathrm{kk}} \searrow \quad \nearrow {\scriptstyle \mathrm{ho}} & \\ & \mathrm{KK} & \end{array}$$

1. $L_h : C^*\mathbf{Alg}^{\mathrm{nu}} \to C^*\mathbf{Alg}^{\mathrm{nu}}_h$ Dwyer-Kan at homotopy equivalences

   - get left exact pointed ∞-category

   - $\mathtt{Map}_{C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A,B) \simeq \ell\mathtt{Hom}(A,B)$ (Anima associated to topological Hom-space)

   - L_h is Schochet exact

2. $\mathbb{C} \to K$ is tensor idempotent in $C^*\mathbf{Alg}^{\mathrm{nu}}_h$

   - left Bousfield localization $L_K := K \otimes - : C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$

   – inverts $A \to K \otimes A$

   - get left semi-additive left-exact ∞-category

   - $L_K \circ L_h$ is Schochet-exact and K-stable

   - $\mathtt{Map}_{L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h}(A,B) \simeq \ell\mathtt{Hom}(A, K \otimes B)$

3. force split exactness:

   - in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$: $A * A \simeq A \oplus A$

- $$\begin{array}{ccccccccc} & & & & & \overset{\iota_0}{\curvearrowleft} & & & \\ 0 & \longrightarrow & qA & \longrightarrow & A * A & \xrightarrow{fold} & A & \longrightarrow & 0 \\ & & & \searrow^{\iota_A} & \downarrow & & & & \\ & & & & A & & & & \end{array}$$

$0 \to qA \to A * A \overset{\nabla}{\to} A \to 0$, $qA \to A$

- $qA \to A$

– any split exact functor should invert ι_A

– Dwyer-Kan localize at $W := \{\iota_A | A \in C^*\mathbf{Alg}^{\mathrm{nu}}\}$

– $L_q : L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h \to L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h[W^{-1}] =: \mathrm{KK}$

– get additive left-exact ∞-category (easy)

– $\mathrm{kk} := L_{q,K,h} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ is homotopy invariant, K-stable and Schochet- and split exact and aninital with this property.

– calculus of fractions: $\mathtt{Map}_{\mathrm{KK}}(A, B) \simeq \mathtt{colim}_{n\in\mathbb{N}}\, \ell\mathtt{Hom}(q^n A, B \otimes K)$

This is a complete description of $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ independent of clasdical cycle by relations picture!

deep theorem of Cuntz: $q^2 A \to qA$ is equivalence in $L_K C^*\mathbf{Alg}^{\mathrm{nu}}_h$

– implies: L_q is right Bousfield localization

– $\mathtt{Map}_{\mathrm{KK}}(A, B) \cong \ell\mathtt{Hom}(qA, B \otimes K)$

– this is one of the classical formulas for KK

stability, fibre sequences ?

interpretation of Cuntz-Skandalis:

Theorem 38.9 (Automatik semiexactess)**.** *For any left-exact additive ∞-category $\mathbf{A}$*

$$\mathrm{incl} : \mathbf{Fun}^{h,K,se+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{A}) \to \mathbf{Fun}^{h,K,split+Sch}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{A})$$

is an equivalence.

apply to kk : $C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$

- conclude: kk is semiexact

- note that $\Omega := \mathrm{kk}(C_0(\mathbb{R})) \otimes -$ is loop endofunctor of left-exact structure

- use Toeplitz extension as above: $\Omega^2 \simeq \mathtt{id}_{\mathrm{KK}}$

Corollary 38.10. KK *is a stable* ∞*-category with bi-leftexact symmetric monoidal structure.*

- gives long exact sequences, Bott periodicity, triangulated structure on hoKK

se $\implies$ *Sch* + *split* if target is stable:

Corollary 38.11. $\mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{KK}$ *is an inital homotopy invariant,* K*-stable and semi-exact functor to a stable* ∞*-category. All works with symmetric monoidal structures* $\otimes_{\max}$ *and* $\otimes_{\min}$*.*

for any stable $\mathbf{C}$

$$\mathrm{kk}^* : \mathbf{Fun}^{ex}(\mathrm{KK}, \mathbf{C}) \xrightarrow{\simeq} \mathbf{Fun}^{h,K,se}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{C})$$

How fits E-theory in this picture?

for exact sequence $0 \to A \to B \to C \to 0$ get comparison map $\mathrm{kk}(A) \to \mathrm{Fib}(\mathrm{kk}(B) \to \mathrm{kk}(C))$

- equivalence if sequence is semisplit, but not in general

Dwyer-Kan W:= all comparison maps

- get functor $c : \mathrm{KK} \to \mathrm{KK}[W^{-1}] =: \mathrm{E}$

- set $\mathrm{e} := c \circ \mathrm{kk} : C^*\mathbf{Alg}^{\mathrm{nu}}\mathrm{KK} \to \mathrm{E}$

Theorem 38.12. $\mathrm{e} : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathrm{E}$ *is a Dwyer-Kan localization and the inital homotopy invariant,* K*-stable and exact functor to a stable* ∞*-category. All works with symmetric monoidal structure* $\otimes_{\max}$*.*

$$\mathrm{e}^* : \mathbf{Fun}^{ex}(\mathrm{E}, \mathbf{C}) \xrightarrow{\simeq} \mathbf{Fun}^{h,K,ex}(C^*\mathbf{Alg}^{\mathrm{nu}}, \mathbf{C})$$

comparsion with classical E-theory (Higson)

- commutes by comparison of universal properties

$$\begin{array}{ccc} C^*\mathbf{Alg}^{\mathrm{nu}} & \xrightarrow{\mathrm{e}_0} & \mathrm{E}_0 \\ & \searrow^{\mathrm{e}} \quad \nearrow_{\mathrm{ho}} & \\ & \mathrm{E} & \end{array}$$

Get lax symmetric monoidal, homotopy invariant, K-stable and exact spectrum-valued K-theory functor for free:

$$K(-) := \mathtt{map}_{\mathrm{E}}(\mathrm{e}(\mathbb{C}), \mathrm{e}(-)) : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Sp}$$

- for homotopy invariant, exact and K-stable $F : C^*\mathbf{Alg}^{\mathrm{nu}} \to \mathbf{Sp}$

– $\mathrm{Nat}(K, F) \simeq \Omega^\infty F(\mathbb{C})$ (Yoneda Lemma)

– $\pi_0\mathrm{Nat}(K, K[1/p]) \cong \mathbb{Z}[1/p]$

application:

- for commutative algebras have $\Psi^p : K \to K[1/p]$ (Adams operation)

- does not extend to all C^*-algebras