Categorification of 1 and of the Alexander pol.

Regensburg, Nov 17. 2020

jt with E. Wagner

arXiv: 1801.02244
1902.05648

I - $\mathfrak{gl}_N$ MOY calculus

II - Categorification.

I –

„A

Res

ass

fix

① $\vec{\Gamma}$

② Γ

①+②

recipe for computing hetikhin-Turaev invariants with $U_q(\mathfrak{gl}_N)$."

$N \in \mathbb{Z}_{\geq 0}$

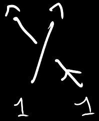
oriented link diagr. $\rightsquigarrow$ $\mathbb{Z}[q,q^{-1}]$ lin comb of graph.

$\rightsquigarrow \langle \Gamma \rangle_N \in \mathbb{Z}_{\geq 0}[q,q^{-1}]$

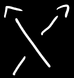
$\rightsquigarrow \langle \vec{D} \rangle_N \in \mathbb{Z}[q,q^{-1}]$

Thm (Reshi

is a link

① 

$\rightsquigarrow$



$\rightsquigarrow$

②

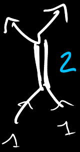
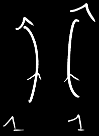
$[n] =$

$[n]! =$

$\left[\begin{matrix} a+b \\ a \end{matrix}\right.$

hiklin Turaev) $\langle \vec{D} \rangle_N$

invariant.

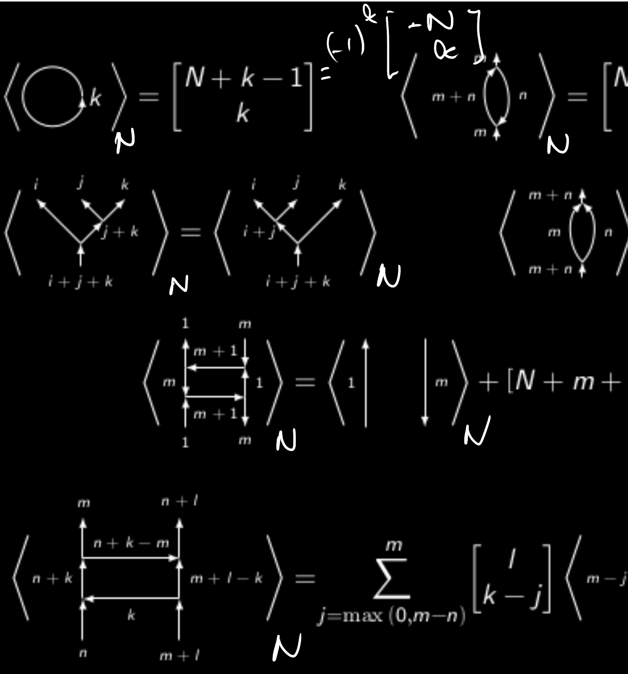
q^{-N} 
$- q^{-N-1}$ 

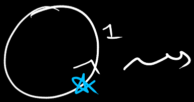
q^{N} " $- q^{N+1}$ "

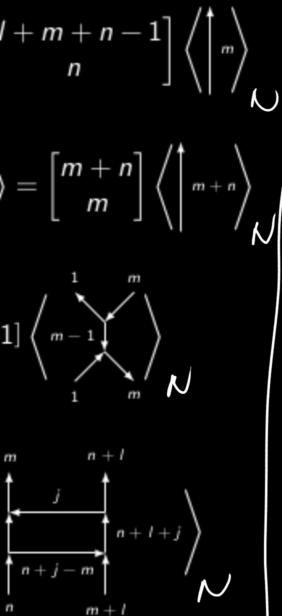
$$\frac{q^n - q^{-n}}{q - q^{-1}}$$

$[1][2]\dots[n]$

$$] = \frac{[a+b]!}{[a]![b]!}$$


$\left\langle \bigcirc k \right\rangle_N = \begin{bmatrix} N+k-1 \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} -N \\ k \end{bmatrix}$
m+n
n
m
N
= [N
i
j
k
j+k
i+j+k
N
=
i
j
k
i+j
i+j+k
N
m+n
m
n
m+n
1
m
m+1
m
1
m+1
1
m
N
=
1
m
N
+ [N + m +
m
n+l
n+k−m
n+k
m+l−k
k
n
m+l
N
$= \sum_{j=\max(0,m-n)}^{m} \begin{bmatrix} l \\ k-j \end{bmatrix}$
m−j




$l+m+n-1$
n
m
N
$= \begin{bmatrix} m+n \\ m \end{bmatrix}$
$m+n$
N
1]
1
m
$m-1$
1
m
N
m
$n+l$
j
$n+l+j$
$n+j-m$
n
$m+l$
N

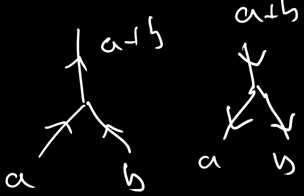
Rmk: consistency is ensured by rep. theory.

Enough to compute Wu

Kauffman-Vogel   Ohtsuki

Murakami Yamada

Def: A MOY graph is a trivalent, $\underbrace{\text{oriented, labeled}}$ (by $\mathbb{Z}_{\geq 0}$), plane graph



"flow condition"

Rmk: graphs arising from step 1 are labeled only by 1 and 2.

× for N=1 and Γ $\{1,2\}$-labeled then

$$\langle \Gamma \rangle_{1} = \underset{\substack{\overset{!}{\shortparallel} \\ q+q^{-1}}}{[2]}^{\#\{ \underset{\wedge}{\overset{\vee}{\|}}{}^{2} \text{ in } \Gamma \}}$$

in g^{al} there is a nice formula for all Γ (but not all N.

× for N=1 $\langle \vec{L} \rangle_{1} = 1$ for any link L

× for N=0 $\langle \Gamma \rangle_{0} = 0$ for any link

because $\langle$

× for N=2

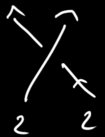
(unnormali

× you get

labelling

labels ~

crossings



$\rightsquigarrow$

$\bigcirc_k \rangle_0 = 0$

$\langle \vec{L} \rangle_2$ is the

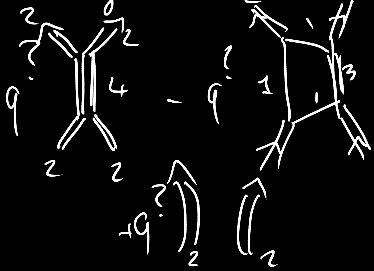
zed) Jones poly.

colored Jones by

strands with different

→ adjust rule for

e.g.

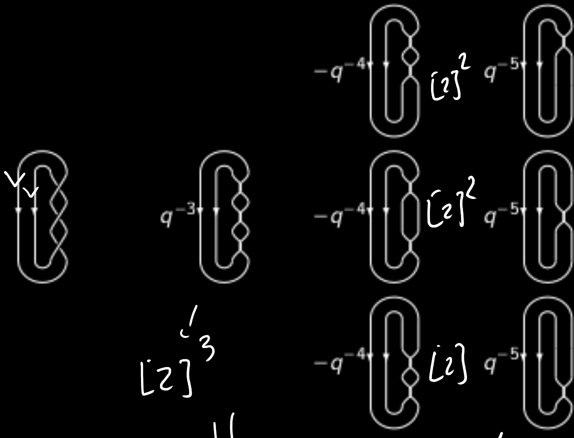


2
2
2
q^?
4
- q^?
1
1
3
2
2
2
+q^?
2
2

To get Alexander, set

put a base point * on

$\langle \bigcirc^{*}_{1} \rangle_0 = 1.$

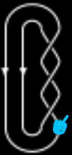


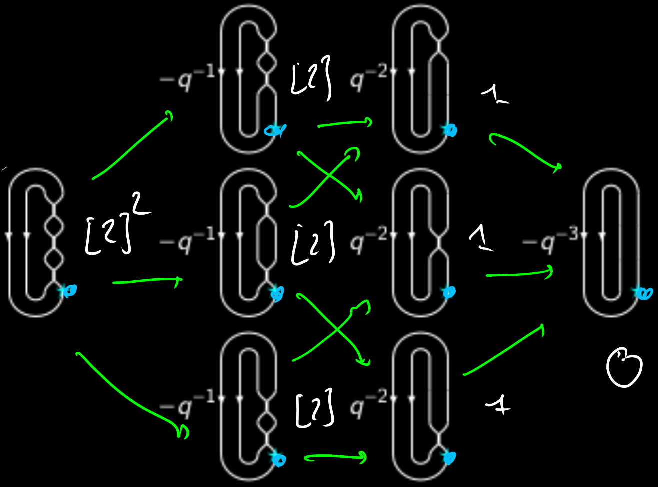
$-q^{-4}$
$[2]^2$
q^{-5}
q^{-3}
$-q^{-4}$
$[2]^2$
q^{-5}
$[2]^3$
$-q^{-4}$
$[2]$
q^{-5}

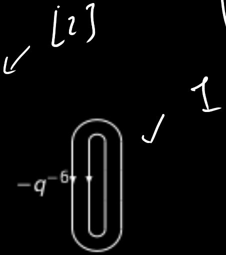
sum everything ⟶ 1

N = 0 +

d define




$[2]^2$
$-q^{-1}$
$[2]$
q^{-2}
1
$-q^{-1}$
$[2]$
q^{-2}
1
$-q^{-3}$
$-q^{-1}$
$[2]$
q^{-2}
1
0



$$[2]^2 - 3[2]q^{-1} + 3q^{-2} = q^2 - 1 + q^{-}$$

II Categorification

Very similar to Khovanov homology

1) hypercube →

2) Cook TQFT-like functor

3) homological algebra

2 ↳ Appear in different forms in papers by Cautis Queffelec-Rose-Sartori.

We focus on

we start with

we want a func

graded

$\Gamma \longmapsto$ ve

→ $\longmapsto$ lin

s.t. $\dim_q F_1(\Gamma)$

First restrict to

⇒ restriction on

with. They are

2) .

$\langle \cdot \rangle_1$

tor F_1

ctor space.

ear maps Γ is $\{1,2\}$-col.

$= \langle \Gamma \rangle_1 \overset{\swarrow}{=} [2]^{\# \{Y\}}.$

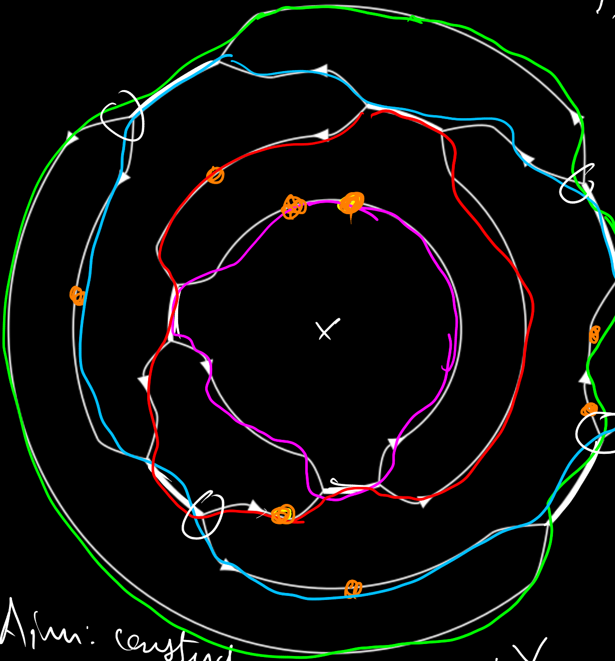
---

braid closure

MOY graph we deal

called vinyl graphs

Today only $\{1,2\}$ colored



Aim: construct

$F_1(\Gamma)$ with dim $[2]^{\# Y} = [2$

index : # tracks of the vinyl
$\overset{\shortparallel}{k}$ here 4

decoration : dots on thin edges

$$\deg(d) = 2\#\bullet - \#$$

$$\underset{\nearrow}{D(\Gamma)} = \bigoplus_{\substack{d \\ \text{dec.}}} \mathbb{Q}_d$$ graded vector space

algebra $\emptyset$-dec is the unit

polynomial algebra (variable ⟷ edges)

(omni)coloring: ① ② ③ ④

pick k circles cover the graph with.

$$\tau(\underset{\text{dec}}{d}, \underset{\text{col}}{c}) = \frac{\prod_{i=1}^{k} X_i^{\#\{\bullet \text{ on } C_i\}}}{\prod_{i \frown j} (X_i - X_j)} \qquad \frac{X_1^2 Y_2^2 X_3^2 X_4^2}{(X_2-X_3)(Y_3-Y_2)^2 (X_4-X_3) \ldots}$$

$\tau(d)$

Prop :

$(d, d'$

bil m

$D(\Gamma$

$F_1($

Prop

$$= \sum_{c} \tau(d,c)$$

$$\tau(d) \in \mathbb{Z}[X_1, \ldots X_k]^{S_k}$$

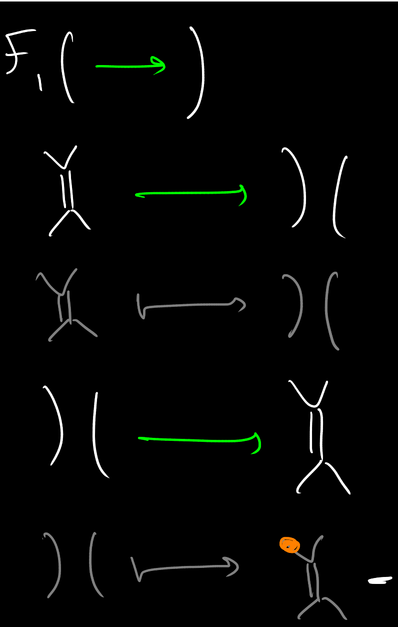
$$) = \tau(dd')\big|_{X_i \mapsto 0} \in \mathbb{Z}$$

ap on

)

$$\Gamma) = \mathcal{D}(\Gamma) \Big/ \mathrm{Ker}((-,-))$$

$$: \dim_q F_1(\Gamma) = [2]^{\#\{\pi\}} = \langle \Gamma \rangle_1$$



Thm ('18): F_1 + classical (step 1) and 3). gl(

homo. link invariant which categorifies 1.

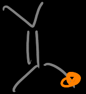
(if not the trivial invariant).

For alexander

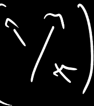
$F_0(\Gamma_*) \subseteq F_1(\Gamma)$

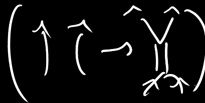
generated by dec with at least

$k-1$ ● next to $*$.



Analogue of holds for knots



$=$ Cone