HIOB: Modular representation theory

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1 Introduction

In this seminar we will introduce basic techniques for studying representations of a finite group G over a field k. We will start the seminar by discussing briefly the theory in the non-describing characteristic case (char k not dividing $|G|$) and then focus to the modular case (char of k dividing $|G|$) which is usually less known. Representations over a field of nonzero characteristic arise in many parts of mathematics:

- In number theory, groups arise as Galois groups of field extensions, giving rise not only to representations over the ground field, but also to integral representations over rings of integers (in case the fields are number fields). It is natural to reduce these representations modulo a prime ideal, at which point we have modular representations.
- In topology, a group may act as a group of self-equivalences of a topological space. thereby giving representations of the group on the homology groups of the space. If there is torsion in the homology these representations require something other than ordinary character theory to be understood.
- In the theory of error-correcting codes many important codes have a non-trivial symmetry group and are vector spaces over a finite field, thereby providing a representation of the group over that field.

The main reference for this seminar is [2]. Another recommended reference is [1]. The topic is self-contained and does not require much more prerequisite other than basic commutative algebra. If you are interested in giving a talk please send me an email at `luca.pol@ur.de`

2 Talks

2.1 Intro talk (14 Oct)

2.2 Representations and Maschke's theorem (21 Oct)

Present the basic definitions and examples to do with group representations. In particular show that a representation is just a module over the group algebra.

Prove Maschke's theorem, which states that in many circumstances representations are completely reducible. Describe the properties of semisimple modules. Present the Artin-Wedderburn structure theorem for semisimple algebras and its immediate consequences.

References: [2, Chapter 1 and 2].

2.3 Character theory (28 Oct)

Define characters over a field of characteristic zero and discuss the character table of the group (please give some examples). Discuss basic properties of characters: rhe character of a direct sum, tensor product or dual is the sum, product or complex conjugate of the characters. The characters of the simple representations form an orthonormal basis for class functions with respect to a certain bilinear form. The character table is square and satisfies row and column orthogonality relations. The number of rows of the table equals the number of conjugacy classes in the group.

References: [2, Chapter 3] up to section 3.4.

2.4 Theorems of Mackey and Clifford (4 Nov)

Present the Mackey's decomposition formula which is a relationship between induction and restriction. After that explain Clifford's theorem, which shows what happens when a simple representation is restricted to a normal subgroup. Discuss the consequence of Clifford's theorem that simple representations of p-groups are induced from 1-dimensional representations of subgroups.

References: [2, Chapter 5].

2.5 Representations of p-groups in characteristic p (11 Nov)

Describe completely the representations of cyclic p-groups. Show that p-groups have only one simple module in characteristic p. Introduce the radical and socle series of modules and deduce that the regular representation is indecomposable, identifying its radical as the augmentation ideal.

References: [2, Chapter 6].

2.6 Projective modules for finite dimensional algebras (18 Nov)

Discuss projective modules and projective covers. Show that the indecomposable projective modules for a finite dimensional algebra over a field are exactly the projective covers of the simple modules. Each has a unique simple quotient and is a direct summand of the regular representation.

References: [2, Chapter 7].

2.7 Projective modules for group algebras (25 Nov)

Summarize the properties of projective modules for p-groups and also the behavior of projective modules under induction and restriction. Show that the Cartan matrix is symmetric (then the field is algebraically closed) and also that projective modules are injective.

References: [2, Chapter 8].

2.8 Splitting fields and the decomposition map (2 Dec)

Examine the relationship between the representations of a fixed group over different rings. Introduce the notion of a splitting field, showing that such a field may always be chosen to be a finite extension of the prime field. After proving Brauer's theorem, that over a splitting field of characteristic p the number of non-isomorphic simple representations equals the number of conjugacy classes of elements of order prime to p, discuss the question of reducing representations from characteristic 0 to characteristic p. Discuss properties of blocks of defect zero. These are representations in characteristic p that are both simple and projective. They always arise as the reduction modulo p of a simple representations in characteristic zero, and these are also known as blocks of defect 0.

References: [2, Chapter 9]

2.9 Brauer characters (16 Dec)

This is a version of character theory but in characteristic p. Introduce Brauer characters and show that they satisfy similar properties to ordinary characters, except that they are only defined on p-regular elements of G. Show that two Brauer characters if and only if the representations have the same composition factors. Show that the tables of Brauer characters of simple modules and of projective modules satisfy orthogonality relations.

References: [2, Chapter 10]

2.10 Indecomposable modules (13 Jan)

Discuss indecomposable modules, local rings and the Krull-Schmidt theorem. When G has a normal cyclic Sylow p-subgroup, the group algebra is a Nakayama algebra and the indecomposable modules are completely described. Discuss the notion of relative projectivity. In particular show that every module is projective relative to a Sylow p-subgroup (Proposition 11.3.5). References: [2, Chapter 11] Section 11.1, 11.2 and 11.3

2.11 Vertices, source and Green's correspondence (20 Jan)

Discuss the trichotomy: finite, tame and wild representation type. Introduce vertices, source and discuss Green's correspondence.

References: [2, Chapter 11] Section 11.4, 11.5 and 11.6

2.12 Block theory (27 Jan)

Give an introduction to the theory of blocks of the group algebra. As the material is a lot for a single talk, focus on giving definitions and conveying the ideas of the theory rather than on giving proofs.

References: [2, Chapter 12]

References

[1] D. J. Benson, *Representations and cohomology*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1991.

[2] Peter Webb, *A course in finite group representation theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2016.