# Spectral order invariant and obstruction to Stein fillability

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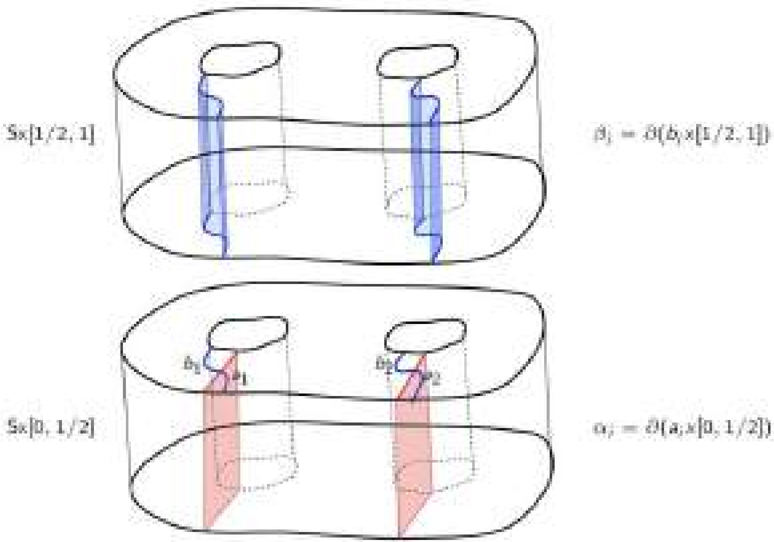
Regensburg Topology, June 16 2020

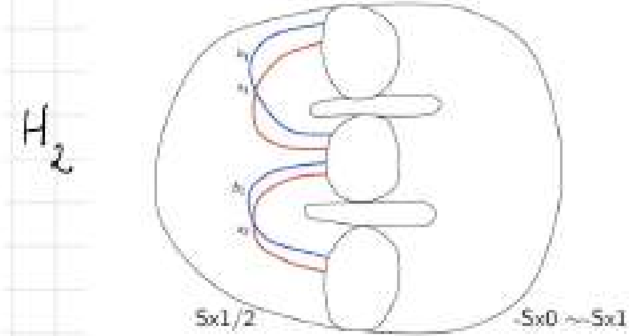
So the open book decomposition $M = S \times [0,1]/\sim_h$ gives us a Heegaard decomposition $M = H_1 \underset{\Sigma}{\cup} H_2$

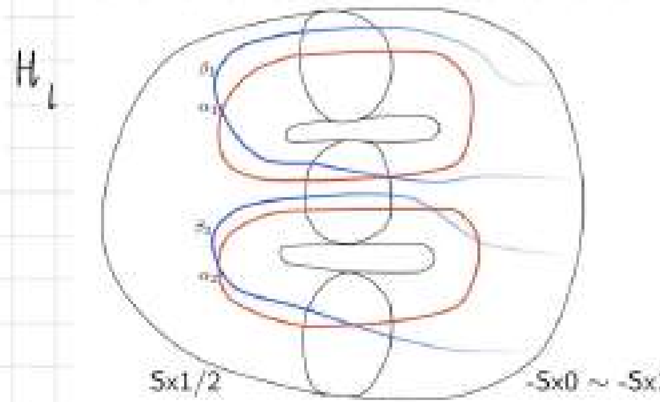
$H_1 = S \times [0, \frac{1}{2}]$, $H_2 = S \times [\frac{1}{2}, 1]$

$\Sigma = \partial H_1 \cup \partial H_2 = \left( S \times \{\frac{1}{2}\} \right) \cup -(S \times \{0\})$ oriented as $\partial H_1$

We also get, given a choice of "basis arcs", $a_1, \dots, a_k \subset S$ that cut $S$ into a disc a natural Heegaard diagram, i.e two families of curves on $\Sigma$, $\underline{\alpha} = \{\alpha_1, \dots, \alpha_k\}$, $\underline{\beta} = \{\beta_1, \dots, \beta_k\}$ each cutting $\Sigma$ into a punctured ball, and $\alpha_i = \partial D_{a_i}$, $\beta_i = \partial D_{b_i}$, $D_{a_i} \subset H_1$, $D_{b_i} \subset H_2$ that cut the handlebodies down to $B^3$.




$(\Sigma, \underline{\alpha}, \underline{\beta})$ is a Heegard diagram for

$$M = M_{(S,h)} = S \times [0,1] / \sim_h$$

# Heegaard Floer Homology (Ozsváth - Szabó)

A Heegaard diagram for $M^3$, $(\Sigma, \underline{\alpha}, \underline{\beta}, z)$ determines a chain complex $CF(\Sigma, \underline{\alpha}, \underline{\beta}, z) \circlearrowleft \partial$ freely generated

$\underline{x} = (x_1, \dots, x_g)$ $\quad x_i \in \alpha_i \cap \beta_{\sigma(i)}$ for some $\sigma \in Sym(g)$.

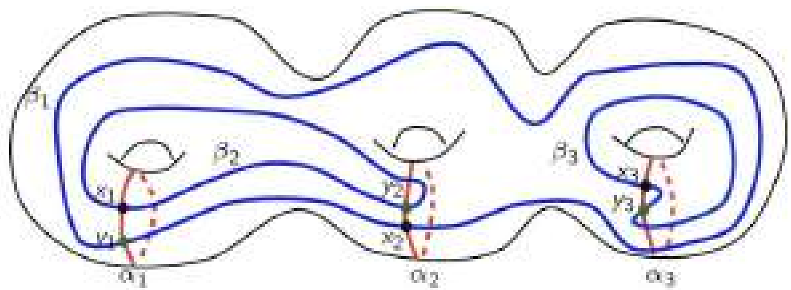
and with $\partial$ determined by counting holomorphic discs in $Sym^g(\Sigma)$. Roughly: $T_\alpha = \alpha_1 \times \cdots \alpha_g$, $T_\beta = \beta_1 \times \cdots \beta_g$

$$(x_1, \dots, x_g) \in T_\alpha \cap T_\beta \subset Sym^g(\Sigma) = \Sigma \times \cdots \times \Sigma /_{Sym(g)}$$

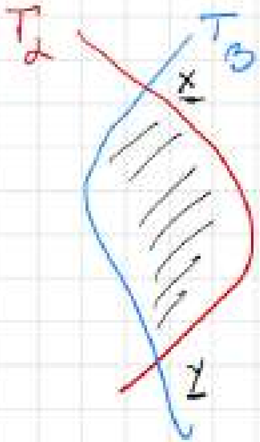
$\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_g)$

$\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_g)$

$\mathbf{x}, \mathbf{y} \in \mathbf{T}_\alpha \cap \mathbf{T}_\beta$



$$\partial_{HF}(\underline{x}) = \sum_{\underline{y}} \sum_{A \in \pi_2(\underline{x},\underline{y})} n(\underline{x},\underline{y},A)\, \underline{y}$$




$\subset \mathrm{Sym}^g(\Sigma)$ (avoiding $\underline{z}$)

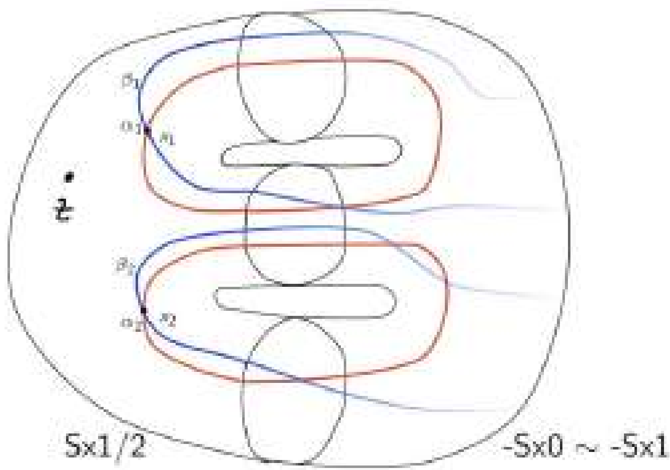
$$\partial_{HF} \circ \partial_{HF} = 0$$

$$\Rightarrow \widehat{HF}(\Sigma, \underline{\alpha}, \underline{\beta}, z)$$

independent of choices : $\widehat{HF}(M)$.

If $M$ carries $\xi$, Ozsvath–Szabo : $c(\xi) \in \widehat{HF}(-M)$

From the point of open book decompositions and the associated Heegaard diagrams:




**Theorem** (Honda-Kazez- Matic): $\mathbf{x} = [(x_1, x_2)] = c(\xi) \in \widehat{HF}$

$\underline{x}$ the contact element has $\partial_{\widehat{HF}} = 0$ because of the placement of $z$ and the requirement that the hol. disks we count in $\partial_{\widehat{HF}}$ miss $z$.

To define $o(\xi)$ we introduce a filtration on $\partial_{\widehat{HF}}$.

Recall

$$\hat{\partial}_{HF}(\underline{x}) = \sum_{\underline{y}} \sum_{\substack{A \in \pi_2(x,y) \\ ind(A)=1}} n(\underline{x}, \underline{y}, A)\, \underline{y}$$

$\pi_2$ relative homotopy classes of maps of punctured disk with the punctures going to $\underline{x}$ and $\underline{y}$

$ind(A)$ = Maslov index of class $A$

$n(\underline{x}, \underline{y})$ a signed count of J-holomorphic curves representing $A$ ( $ind(A)=1 \Rightarrow$ 1-dimensional mod space $\Rightarrow /\mathbb{R}$ translation gives a count )

To filter $\hat{\partial}_{HF}$ we define (following Hutchings idea for ECH)

$$I_+(A) = \mu(D(A)) - 2e(D(A)) + |x| + |y|$$

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$D(A)$ = domain in the Heegard diagram corresponding to $A$, with corners at points of $\underline{x}$ and $\underline{y}$ and boundary on $\{\alpha_i\} \cup \{\beta_i\}$

$\mu(D(A))$ = Maslov index, $e(D(A))$ = Euler measure (following Lipshitz reformulation)

$|\underline{x}|$ = # of cycles in the permutation $\sigma$ where $\underline{x} = (x_1, \ldots, x_g)$ $\quad x_i \in \alpha_i \cap \beta_{\sigma(i)}$

<u>Note:</u> if $\underline{x} = \underline{x}_\xi$, then $x_i \in \alpha_i \cap \beta_i$ so $|\underline{x}_\xi| = g$.

When there is a holomorphic curve representing $A$ we can use

$$J_+(A) = 2\,[n_x(\mathcal{D}(A)) + n_y(\mathcal{D}(A))] - 1 + |x| - |y|$$

where $n_x(D) = \sum_i n_{x_i}(D)$ and $n_{x_i}(D)$ is the local multiplicity of $D$ at the corner $x_i$.

Turns out $J_+(A) = 2\ell$ and we decompose $\hat{\partial}_{HF}$ (following Hutchings idea in ECH)

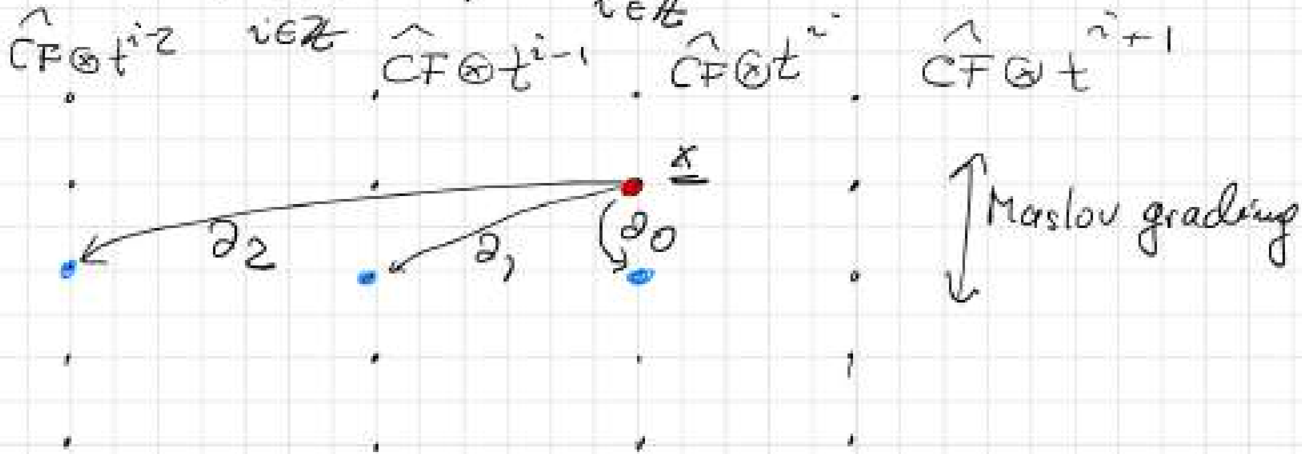
$$\hat{\partial}_{HF} = \partial_0 + \partial_1 + \cdots + \partial_\ell + \cdots$$

$\partial_\ell$ counts $J$-holomorphic curves with $J_+(A) = 2\ell$

<u>More explicitly:</u> Given $(M,\xi)$ and a compatible open book $(S,h)$ with a basis of arcs $\underline{a}=(a_1,\dots,a_g)$ and the corresponding Heegaard diagram $(\Sigma,\underline{\alpha},\underline{\beta},z)$ form

$$\widehat{\mathcal{CF}}(S,h) \overset{\text{def}}{=} \widehat{CF}(\Sigma,\beta,\alpha,z)\otimes_{\mathbb{F}}\mathbb{F}[t,t^{-1}]$$

and define
$$\hat{\partial}\Big(\sum_{i\in\mathbb{Z}} c_i t^i\Big) = \sum_{i\in\mathbb{Z}}(\partial_\ell(c_i))\,t^{i-\ell}$$



$$\sum_{i+j=l} \partial_i \circ \partial_j = 0 \quad \text{because of additivity of } \mathcal{J}_+$$

$\widehat{CF}(S, h, \underline{a})$ is a filtered chain complex

$$\mathcal{F}^p(S, h, \underline{a}) = \left\{ \sum_{i \leq p} c_i t^i \;\middle|\; c_i \in \widehat{CF}(\Sigma, \beta, \alpha) \right\}$$

$\Rightarrow$ Spectral sequence

<u>Define:</u> $\sigma(S, h, \underline{a})$ = smallest non-negative $k$ such that $[x_\xi] = 0$ in $E^{k+1}(S, \phi, \underline{a})$

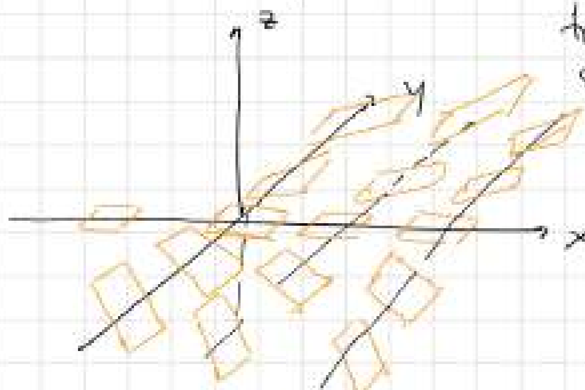
<u>Problem:</u> this might depend on choices

Contact structure on $M^3$ closed, oriented 3-manifold is a plane field $\xi$, $\xi_p \subset T_pM$ that is nowhere integrable: $\xi_p = \ker \alpha$ s.t. $\alpha \wedge d\alpha = dvol$

<u>Examples</u>: standard contact structure on $\mathbb{R}^3$

$\xi_{std} = \ker(\alpha_{std})$, $\alpha_{std} = dz - y\,dx$

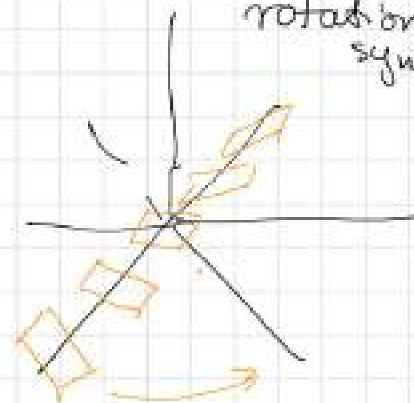
transl. sym.



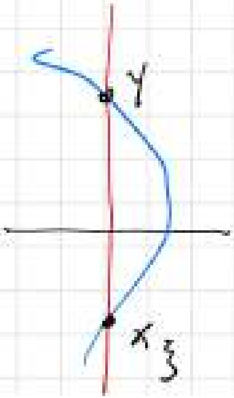

$(r, \theta, z)$

$\alpha = dz + r^2\, d\theta$

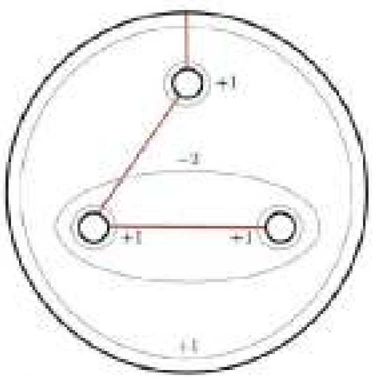
rotational symmetry

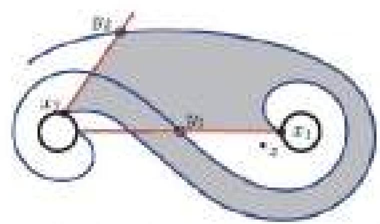


<u>Example:</u> In an overtwisted contact structure, there is always a choice of $(S, h, a)$ so that an $a_i$ is "taken to the left" hence producing a generator that has $\partial(y) = x_\xi$




$\partial S \quad \Rightarrow \quad o(S,h,a) = 0$





- overtwisted contact structure
- only a $J_+ = 2$ domain giving $\partial y = x_\xi$
- $o(S,h,a) = 1$

Define: $o(M,\xi) = \min\{(o(S,\phi,a), J_{HF})\}$

- independence of almost $\propto$ structure and isotopy OK
- We can do this with a a set of arcs containing a basis and multi-pointed diagrams, and keep the same definition
- Main technical result: "triangle ellimination"

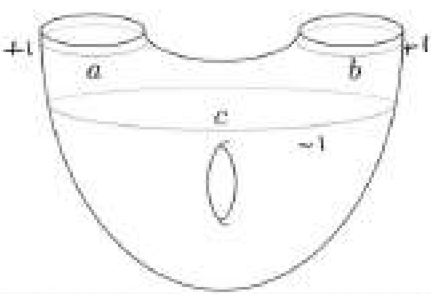

$$o(S,\phi,a) \geq o(S,\phi,a')$$

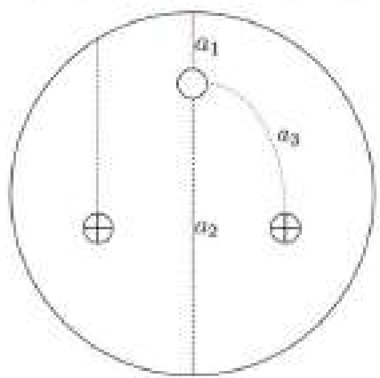
- Legendrian surgery:

  If $(M', \xi')$ is obtained from $(M, \xi)$ by Legendrian surgery then $o(M', \xi') \geq o(M, \xi)$

$\Rightarrow$ <u>Corollary:</u> If $(M, \xi)$ is Stein fillable then $o(M, \xi) = \infty$

<u>Examples:</u> (Conway '19) a family of contact manifolds and open books:




First example $h = \tau_a \tau_b \tau_c^{-1}$

$c(\xi) \neq 0$

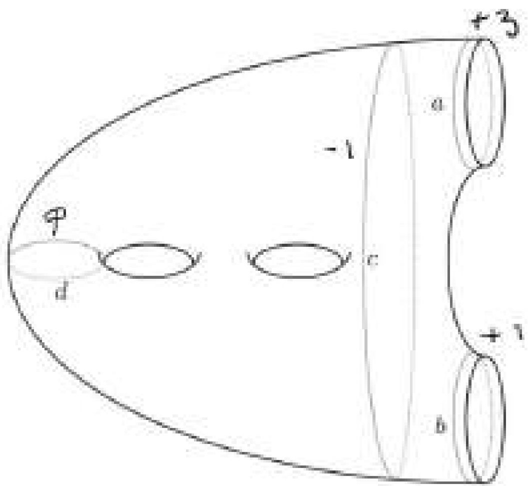
(Conway, Hedden–Plamenevskaya)

There is also a family $(Y_p, \xi_p)_{p \in \mathbb{Z}_{>0}}$

with page $S_{2,2}$ = genus 2 surface with 2 bdy components

$$h_p = \tau_a^3 \tau_b \tau_c^{-1} \tau_d^p$$

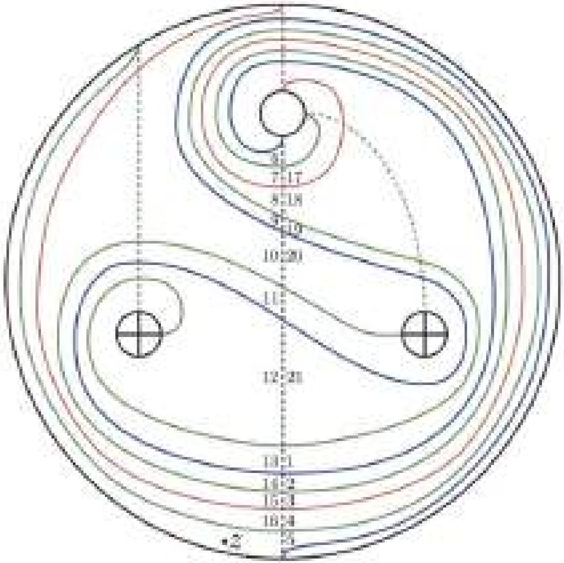
$$c(\xi_{(S_{2,2}, h_p)}) \neq 0 \quad \text{but} \quad \sigma(\xi_{(S_{2,2}, h_p)}) = 0$$

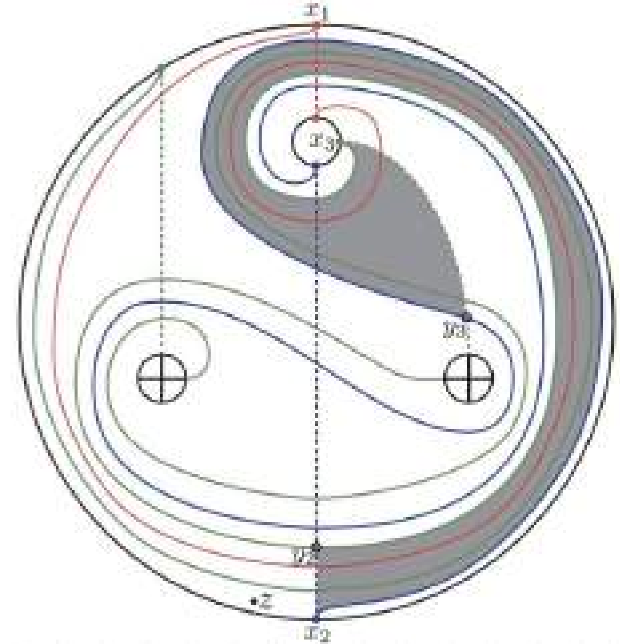


<u>Theorem:</u> $c(Y, \xi) = 0$, hence $(Y, \xi)$ is not fillable.

<u>Lemma:</u> A Maslov index 1 domain $D$ from $x$ to $y$ has $J_+ = 0$ only if it is an immersed $2k$-gon with no corners in the interior. If $|x| - |y| = 1 - k$, then it is an iff statement.

This lemma lets us do the calculation:




open books page diagram for our example.

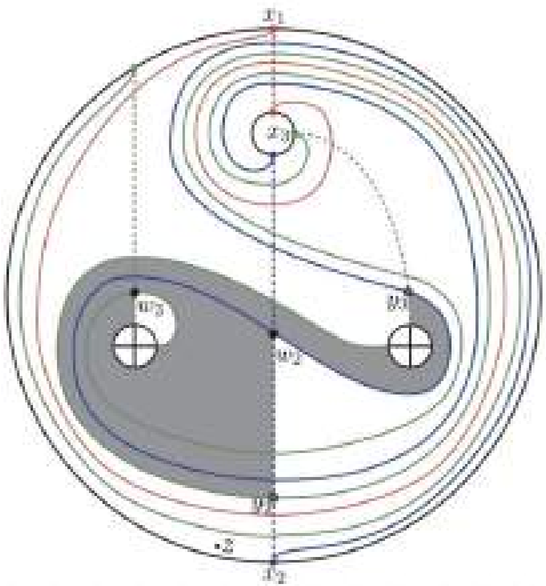
$$J_+(D_1) = 2\left(\tfrac{1}{2}+\tfrac{1}{2}\right) - 1 + 2 - 3 = 0$$

$$\partial_0(x_1, y_2, y_3) = (x_1, x_2, x_3)$$

But $\partial_{HF}(x_1, y_2, y_3) = \partial_0 + \partial_1 =$

$$= (x_1, x_2, x_3) + (x_1, w_2, w_3)$$

with $\partial_1$ represented by:




$$J_+ = 2(n_y + n_w) - 1 + |w| - |y|$$

$$= 2\left(\frac{1}{2} + \frac{3}{2}\right) - 1 + 2 - 3$$

$$= 4 - 1 + 2 - 3 = 2$$

No other domains leave $(x_1, y_2, y_3)$.

Theorem (Darboux) Contact structures are homogeneous, locally they all look like $\xi_{std}$

Theorem (Martinet 1970s) Every 3-manifold supports a contact structure

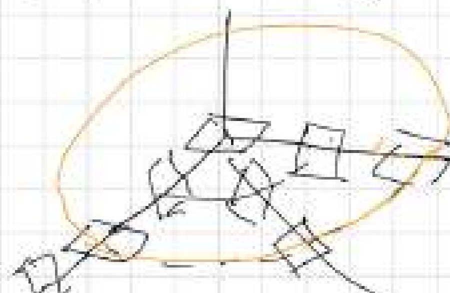
Theorem (Lutz 1970's) Every homotopy class of plane fields supports a contact structure

Q: Are all $\xi$ on $\mathbb{R}^3$ contactomorphic?

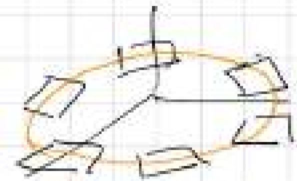
Answer: (Bennequin 1980's) No, $\xi_{st}$ and $\xi = \ker \alpha$

$$\alpha = \cos r \, dz + r \sin r \, d\theta$$

$$\ker \alpha: \quad \frac{dz}{d\theta} = -r \frac{\sin r}{\cos r} = -r \tan r$$



A disk $D^2$ embedded in contact $(M,\xi)$ is <u>overtwisted</u> if at $p\in\partial D^2$, $\xi_p = T_pD \subset T_pM$



$\xi_{OT}$

<u>Theorem (Bennequin, 1983)</u>

Standard contact structure $\xi_t$ on $\mathbb{R}^3$ does not contain an overtwisted disk, so is not contactomorphic to $\xi_{OT}$

Contact structures

<u>tight</u>

<u>E:</u> Only finitely many htpy classes of plane fields support tight contact structures

<u>overtwisted</u>

<u>Eliashberg</u>: classified up to isotopy by the htpy class of the plane field

Eliashberg, Giroux, Lisca-M, ... $\exists$ tight cont-structures htpic as plane fields, not is- as cont-str.

Eliashberg, Gromov: · (Stein (sympl.)) fillable contact structures
$(M, \xi)$, i.e. $\xi = T_pM \cap JT_pM)$ – complex lines in the boundary $M$ of a Stein domain $(W, J)$ are tight.

$\xi$ is (weakly) symplectically fillable if $M = \partial W$, $(W, \omega)$ is a symplectic manifold and $\omega|_\xi > 0$

if $\xi = \ker i_Y \omega$ for some vector field $Y$ pointing transversally outward with $L_Y \omega = \omega$ (Liouville vector field)

If $Y$ is defined on all of $W$, then $i_Y \omega$ is a primitive for $\omega$ and $(W, \omega)$ is exact

Q: how can we recognize tight/fillable vs overtwisted contact structures?

- find OT disk - cut/paste methods
  - open book characterization

- invariants

<u>Ozsváth - Szabó</u> contact invariant in Heegaard Floer homology

Given a contact 3-manifold $(M, \xi)$ there is a distinguished element $c(\xi) \in \widehat{HF}(-M)$ such that
- $c(\xi) = 0$ if $\xi$ is OT
- $c(\xi) \neq 0$ if $\xi$ is Stein fillable

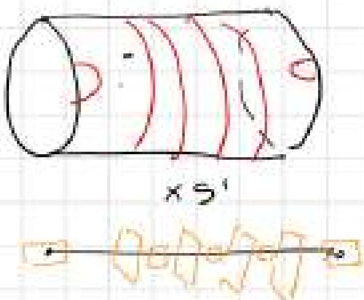
<u>Hope:</u> $c(\xi)$ detects tightness, i.e.

$$c(\xi)=0 \Rightarrow \xi \text{ OT}$$

<u>No:</u> $\exists$ tight contact structures with $c(\xi)=0$

<u>examples:</u> • Ghiggini examples containing Giroux torsion

$(T^2\times I, \xi_n) = (A\times S^1, \xi_n)$




• Honda-Kazez-M more general examples of $S^1$-invariant $\xi$ on $\Sigma\times S^1$, $\partial\Sigma\neq\emptyset$

<u>Q:</u> Is $c(\ )\neq 0$ characterizing fillability?

(strong fillability for $c(\xi)$
weak for $c(\xi)$ with coefficients

A: NO Examples use ad hoc methods, often using cobordism arguments.

Theorem: (Kutluhan - M - VanHorn-Morris - Wand)

There is an invariant $o(M,\xi) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ such that

- $o(M,\xi) = 0$ if $(M,\xi)$ is overtwisted
- $o(M,\xi) = \infty$ if $(M,\xi)$ is Stein fillable
- $o(M,\xi)$ can be detected on any open book decomposition of $M$ compatible with the contact structure $\xi$.

Note: There are $\xi$ with $c(\xi) \neq 0$ but $o(\xi) < \infty$ thus $o(\xi)$ gives new obstruction to Stein fillability.

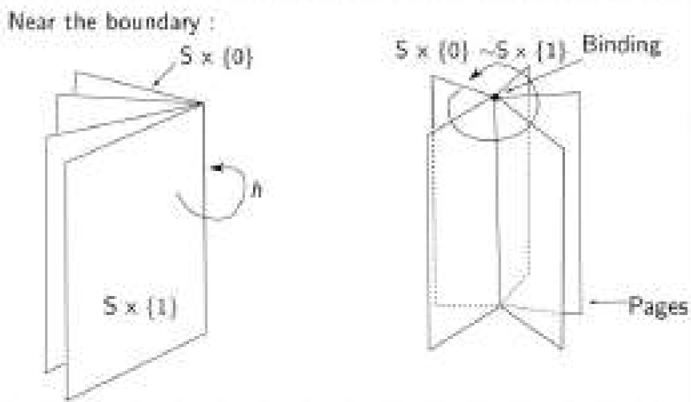
Open book decomposition:

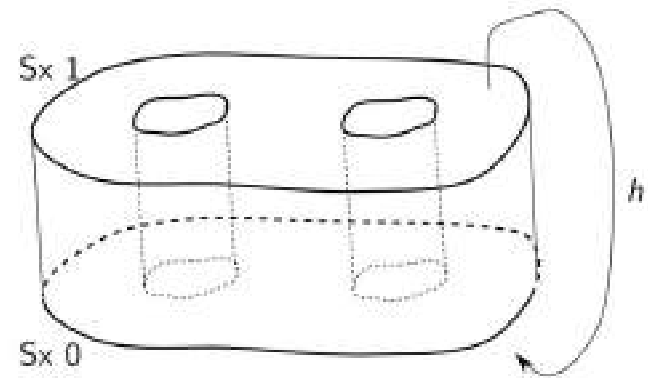
$M^3$, $B^1 \hookrightarrow M$ the binding, $M \cong S \times [0,1] /_{\sim h}$, $h \in \mathrm{Map}(S, \partial S)$

$$\begin{tikzcd} M - B^1 & S_t = \pi^{-1}(t) \arrow[l, hook'] \arrow[d] \\ S^1 \arrow[from=1-1, to=2-1] & t \arrow[l, phantom, "\ni"] \end{tikzcd}$$

$(x,1) \sim_h (h(x), 0)$, $\forall x$

$(x,t) \sim_h (x,t')$, $\forall t,t'$, $\forall x \in \partial S$




$(M, B, \pi)$ is compatible with $\xi$ if $\xi$ is almost tangent to pages $S_t$

Giroux: $\{\text{contact structures on } M\}/\text{isotopy} \overset{1-1}{\longleftrightarrow} \{\text{open book decompositions of } M\}/\text{positive stabilization}$

Open books → Heegaard decompositions:

$M = H_1 \cup H_2$ , $H_i$: handlebodies:

$\Sigma = \partial H_1 = \partial H_2$

compressing discs.

Note: if $S$ is surface with boundary, $S \times [0,1]$ is a handlebody, compressing discs: $a \times [0,1]$, $a \subset S$ properly embedded