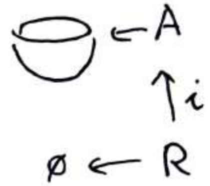
From $SL(2)$ to $GL(N)$ foam evaluation

M. Khovanov

based on joint work with Louis-Hadrien Robert (arxiv 2020),
work of L.-H. Robert and Emmanuel Wagner (arxiv 2017-2018),
joint work with Nitu Kitchloo (arxiv 2020),
joint work with N. Kitchloo & Yakov Kononov (in progress)
+ idea of Y. Kononov for non-closed surface evaluation

Simplest example of link homology (Khovanov ($SL(2)$ homology) is based on a rank 2 commutative Frobenius algebra

$R = \mathbb{Z} \quad A = \mathbb{Z}[X]/(X^2)$, $\varepsilon: \begin{matrix} X \to 1_R \\ 1_A \to 0 \end{matrix}$, $\begin{matrix} \Delta(1) = 1 \otimes X + X \otimes 1 \\ \Delta(X) = X \otimes X \end{matrix}$

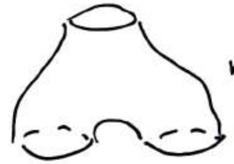


$\varepsilon: A \to R$

Equivariant extensions (Bar-Natan, MK)

$R = \mathbb{Z}[E_1, E_2]$ $\qquad R \cong H^*_{U(2)}(p, \mathbb{Z})$ point

$A = R[X]/(X^2 - E_1 X + E_2)$ $\qquad \cong H^*(BU(2), \mathbb{Z})$



m

Δ

ε - same as before $\qquad A \cong H^*_{U(2)}(\mathbb{P}^1, \mathbb{Z})$

$\Delta(1) = X \otimes 1 + 1 \otimes X - E_1\, 1 \otimes 1$

$\Delta(X) = X \otimes X - E_2\, 1 \otimes 1$

$E_1 = x_1 + x_2$ $\qquad$ E's are symm f's in x's.

$E_2 = x_1 x_2$

$X^2 - E_1 X + E_2 = (x - x_1)(x - x_2)$

Recent variation (L.-H. Robert, MK arXiv 2020)

$R = \mathbb{Z}[x_1, x_2]$ (paper uses α_1, α_2) $\qquad R \cong H^*_{U(1)\times U(1)}(p, \mathbb{Z})$ $\qquad U(1)\times U(1) \subset U(2)$

$A = R[x]/((x - x_1)(x - x_2))$ $\qquad (x - x_1)(x - x_2) = 0$

zero divisors in A

$\Delta(1) = (x - x_1) \otimes 1 + 1 \otimes (x - x_2)$

$\Delta(x - x_i) = (x - x_i) \otimes (x - x_i)$

$(x - x_1)^2 = (x_2 - x_1)(x - x_1)$

$x - x_1,\ x - x_2$ - 'almost' orthogonal idempotents $\xrightarrow[\pm(x_1 - x_2)]{\text{divide by}}$ get idempotents, E. S. Lee's theory

Rasmussen's invariant.

Link homology extends to link cobordisms

M. Jacobsson

D. Bar-Natan

MK

Sign indeterminancy

S. Morrison, K. Walker, D. Clark

C. Caprau

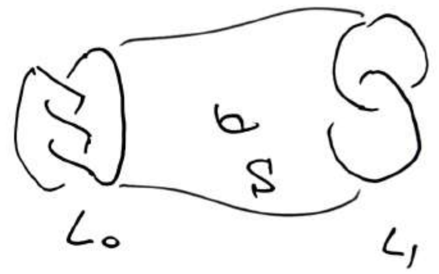
P. Vogel

$\Big\}$ refine cobordisms by introducing defect lines to associate a well-defined map to a cobordism, not up to an overall sign

functor

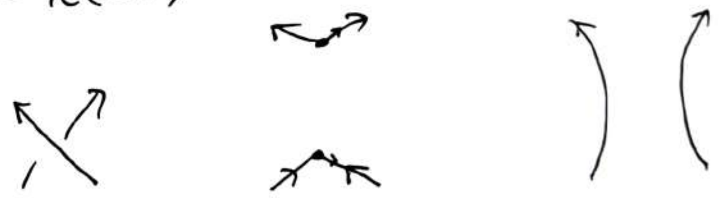
link cobordisms $\xrightarrow{H}$ bigraded R-modules

(R - ground ring)

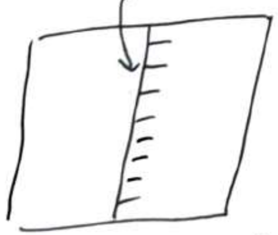


$$H(L_0) \xrightarrow{H(S)} H(L_1)$$

$$\deg H(S) = (0, -\chi(S))$$

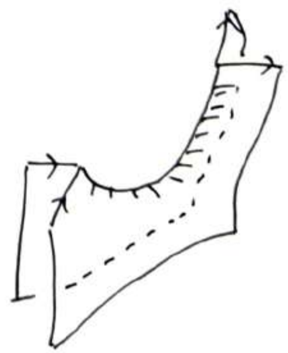


seam line (circle)



co-orientation

(choice of normal direction)



If U surface with boundary, denote by $\theta(U)$ the number of boundary circles of U

$$\chi(\overline{U}) = \chi(U) + \theta(U)$$

$$\overline{U} = U \cup D_1^2 \cup \ldots \cup D^2_{\theta(U)}$$

$\uparrow$

close U by capping off the disks

$\Big\} \; \overline{U}$

Y. Kororov

$\wr$

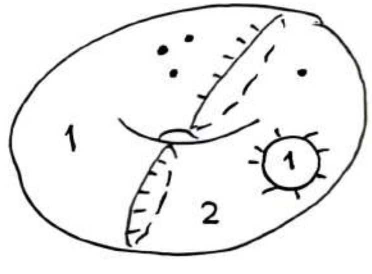
extend eval f-la to surfaces with boundary

$S \subset \mathbb{R}^3$ closed seamed dotted surface embedded in $\mathbb{R}^3$

orientable

c checkerboard coloring of S by $\{1, 2\}$

S connected $\rightarrow$ 2 checkerboard colorings



since seams point into facets, get induced coloring of seams

$\theta_i(c)$ # of seams colored i

$d_i(c)$ # of dots on facets colored i

$S_i(c)$ or $F_i(c)$ union of facets colored i

$\overline{S}_i(c)$ or $\overline{F}_i(c)$ closure of $F_i(c)$

seamed torus

3 seams, 4 dots

coloring c_1

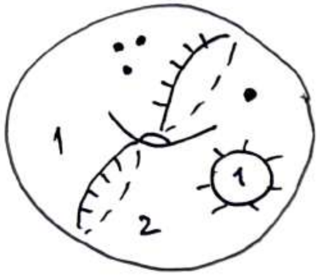
$d_1(c_1) = 3$, $d_2(c_1) = 1$

$\theta_1(c_1) = 1$, $\theta_2(c_1) = 2$

$$s(S, c) = \theta_1(c) + \frac{\chi(\overline{S}_1)}{2} \qquad \langle S, c \rangle = (-1)^{s(S,c)} \frac{x_1^{d_1(c)} x_2^{d_2(c)}}{(x_2 - x_1)^{\chi(S)/2}} \qquad \langle S \rangle = \sum_c \langle S, c \rangle$$

$$\langle S, c\rangle = (-1)^{s(S,c)} \frac{x_1^{d_1(c)} x_2^{d_2(c)}}{(x_2-x_1)^{\chi(S)/2}} \qquad s(S,c) = \theta_1(c) + \frac{\chi(\bar{S}_1(c))}{2}$$

$$\langle S\rangle = \sum_c \langle S, c\rangle$$



$d_1(c_1)=3,\ d_2(c_1)=1$

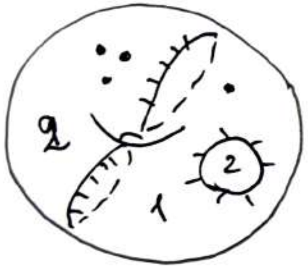
$\theta_1(c_1)=1,\ \theta_2(c_1)=2$

$S_1(c_1)$ 

$s(S,c_1) = 1+2=3$

$\bar{S}_1(c_1)$  $= S^2 \cup S^2 \qquad \frac{\chi(\bar{S}_1(c_1))}{2} = 2$

$$\langle S, c_1\rangle = (-1)^3 \frac{x_1^3 x_2}{(x_2-x_1)^0} = \underline{-x_1^3 x_2}$$



$d_1(c_2)=1,\ d_2(c_1)=3$ $\qquad S_1(c_2)$  $\qquad \bar{S}_1(c_2) =$  $\qquad \frac{\chi(\bar{S}_1(c_2))}{2} = 1$

$\theta_1(c_2)=2,\ \theta_2(c_1)=1$

$s(S,c_2) = 2+1=3$

$$\langle S, c_2\rangle = \frac{(-1)^3 x_1 x_2^3}{(x_2-x_1)^0} = \underline{-x_1x_2^3}$$

$$\langle S\rangle = \langle S, c_1\rangle + \langle S, c_2\rangle = -x_1x_2(x_1^2+x_2^2)$$ – symmetric f'n in x_1, x_2

genus 0 example, so don't see denominator in individual terms

$$\langle S, c\rangle = (-1)^{s(S,c)} \frac{x_1^{d_1(c)} x_2^{d_2(c)}}{(x_2 - x_1)^{\chi(S)/2}} \qquad s(S,c) = \theta_1(c) + \frac{\chi(\bar{S}_1(c))}{2}$$

S

c_1

$\theta_1(c_1) = 1 \quad \bar{S}_1(c_1) = S^2 \quad \chi = 2$

$s(S, c_1) = 1 + \frac{2}{2} = 2$

$$\langle S, c_1\rangle = (-1)^2 \frac{x_1^2}{(x_2 - x_1)}$$

c_2

$\theta_1(c_2) = 0 \quad \bar{S}_1(c_1) = S^2 \quad \chi = 2$

$s(S, c_2) = 0 + \frac{2}{2} = 1$

$$\langle S, c_2\rangle = (-1) \frac{x_2^2}{x_2 - x_1}$$

$$\langle S\rangle = \frac{x_1^2 - x_2^2}{x_1 - x_2} = x_1 + x_2$$

$$\begin{array}{ccc} R & & R' \\ \parallel & & \parallel \\ \mathbb{Z}[E_1, E_2] & \subset & \mathbb{Z}[x_1, x_2] \end{array}$$

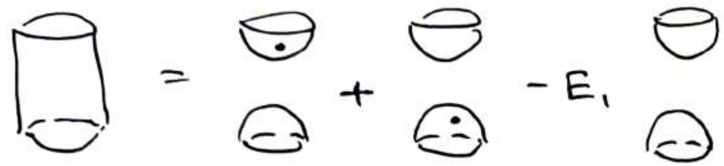
Thm $\langle S\rangle \in R$, for any embedded, seamed, dotted S.

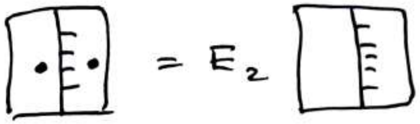
$\langle S \sqcup S'\rangle = \langle S\rangle\langle S'\rangle$

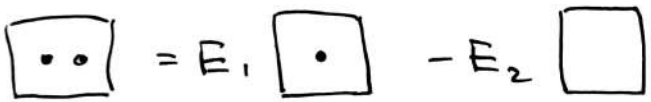
Proof: reduce to connected S, 2 colorings, take the sum.

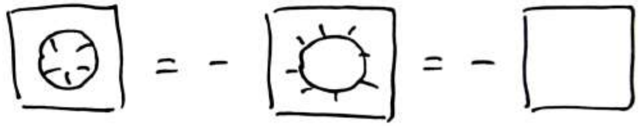
Simple version of Kempe moves of colorings.

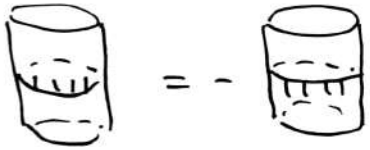
Skein relations



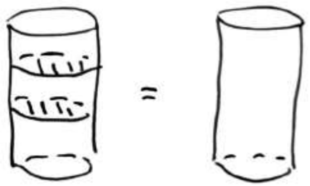






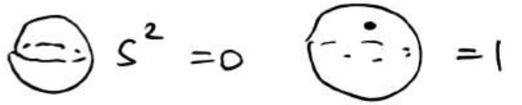


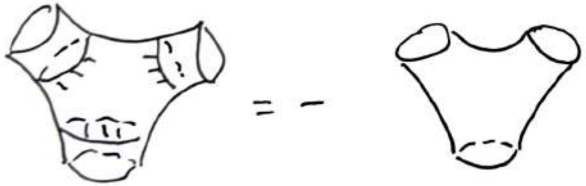
reversing co-orientation



seam circle is like an involution

not a seam





no checkerboard coloring.

seams need to represent 0 in $H_1(S, \mathbb{Z}/2)$, closed S, otherwise no colorings & $\langle S \rangle = 0$

Universal construction

R - commutative ring

f: $\begin{matrix}\text{isom. classes of}\\ n\text{-dim objects}\end{matrix} \longrightarrow R$

$\Leftarrow$

N - closed oriented $(n-1)$-manifold

$Fr(N)$ - free R-module, basis

$[M]$, $\partial M \simeq N$

R-bilin form on $Fr(N)$

$$([M_1],[M_2])_N = f((-M_1)\cup_N M_2)$$

$$f(N) = Fr(N)/\ker((\,,\,)_N)$$

functoriality: cobordism M induces a map

$$f(N_0) \xrightarrow{[M]} f(N_1)$$

$$[M_0] \longmapsto [MM_0]$$

compose cobordisms

$\partial M_0 = N_0$.

Example: $f: M \longmapsto f(M) \in R$

M - smooth oriented closed n-manifold

$$f(M_1 \amalg M_2) = f(M_1)\,f(M_2)$$

multiplicative

$M \simeq M'$ diffeomorphic $\quad f(M) = f(M')$ function on diffeomorphism classes

$\longrightarrow f(M) \in R$

add invol. ψ on R s.t. $f(-M) = \psi(f(M))$

$$f(N_0) \otimes_R f(N_1) \longrightarrow f(N_0 \amalg N_1)$$

not isomorphisms, in general.

isomorphisms in many favorable cases.

functor: n-cobordisms $\longrightarrow$ R-modules.

Universal construction goes back to

Blanchet - Habegger - Masbaum - Vogel TQFTs derived from Kauffman bracket, 1995

WRT TQFTs

used in context of foams in $\mathbb{R}^3$

M.K $sl(3)$ - homology, M. Mackaay, P. Vaz universal $sl(3)$ homology

+ other papers on foam evaluation.

Similar construction from positivity of $(\,,\,)$ viewpoint 2D-4D case

Freedman - Kitaev - Nayak - Slingerland - Walker - Wang Universal manifold pairings and positivity 2005

for introduction & more references, see

K. Walker, Universal manifold pairings in dim 3, Celebratio Mathematica, 2012

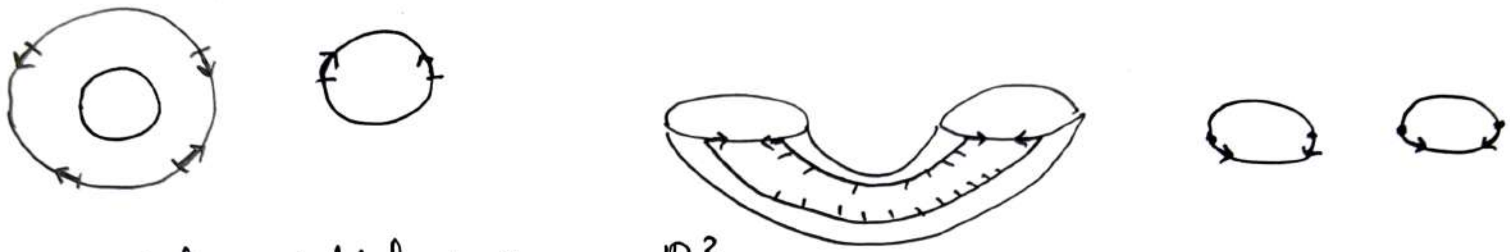
Universal construction for foam evaluation:

L.-H. Robert, E. Wagner, 3 papers, arXiv 2017+ ; many applications

$SL(3)$ case L.H. Robert, MK arXiv 2018, relation to Kronheimer-Mrowka homology for graphs, 4-color theorem

More basic examples of universal constructions (MK, to appear very soon).

$S \subset \mathbb{R}^3$ closed, seamed, dotted surface. Take generic cross-section by a plane $\mathbb{R}^2$
circles in cross-section, marked points. Orientations at points



C marked embedded circles in $\mathbb{R}^2$

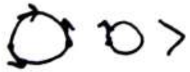
C a <u>mem</u> (marked embedded 1-manifold)

State space of C $\langle C \rangle$ - all marked surfaces S, $\partial S = C$, modulo universal relations.

Examples: $C =$  no marks $\langle C \rangle \cong A \cong R\cdot 1 \oplus R\cdot x$  basis

$C =$  $\langle C \rangle = 0$ not a boundary. need balancing for marked points

each circle must carry even # of marks,
otherwise no colorings.

$\langle$  $\rangle = 0$

 ok

but no colorings. 

<u>various balancing/parity conditions.</u>

Best-case scenario

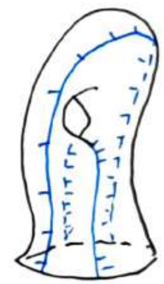
each circle is <u>balanced</u> (2k marks, k oriented clockwise, k anticlockwise)

Even for balanced circles, state space depends on signs sequence

2 Dim

basis

basis



checkerboard
no coloring

Unfinished: determine basis & dim of state space for any planar mem C.

Easy escape: add $\omega = \sqrt{-1}$ to R, redefine

no bar

$$\langle S, c\rangle_\omega = (-1)^{\theta_1(c)}\, \omega^{\chi(S_1(c))} \frac{x_1^{d_1(c)} x_2^{d_2(c)}}{(x_2 - x_1)^{\chi(S)/2}} = \omega^{-\theta(S)} \langle S, c\rangle.$$

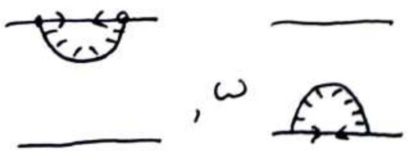
Then $\langle C\rangle_\omega \cong A$ if C - balanced circle

$\langle C\rangle_\omega \cong A^{\otimes k}$ non-canonically

$\theta(S)$ # of seam circles

if C is a union (may be nested) of k balanced circles

ω present in C. Caprau, S. Morrison - D. Clark - K. Walker.



mutually-inverse isomorphisms

SL(2) $\to$ GL(2) foams

C. Blanchet

M. Ehrig, D. Tubbenhauer, C. Stroppel

A. Beliakova, M. Hogancamp, K. Putyra & S. Wehrli

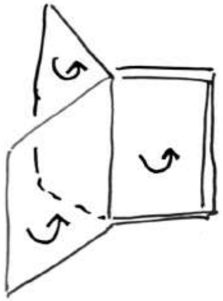
N. Kitchloo, M.K.

foam $F \subset \mathbb{R}^3$

facets of thickness 1 & 2

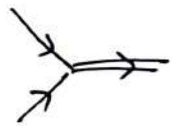
dots

orientations on facets

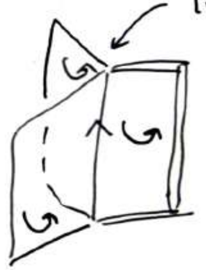


facets compatibly oriented

In one dimension

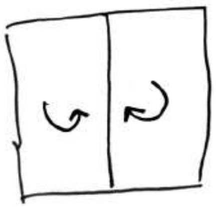


induce orientation on seams



if remove double facet

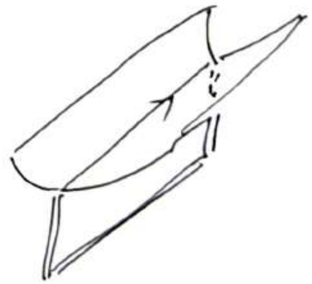
seamed circles



orientation reversed along seam line

second

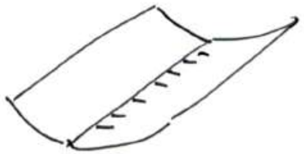
first



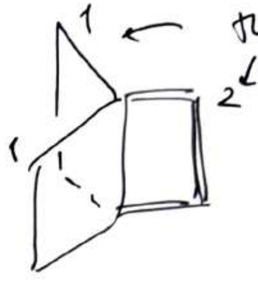
double facet

go counterclockwise

SL(2) foam from GL(2) foam by removing double facets $\Leftarrow$ get co-orientation along seam line



evaluation of $GL(2)$ foams

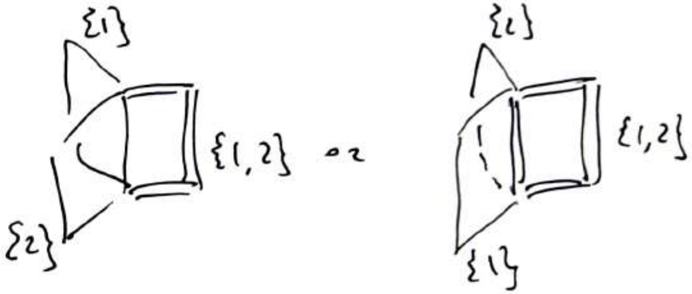


colorings c: facets $\to$ subsets of $\{1,2\}$

thin facet $\to$ subset of cardinality 1

thick facet $\to$ subset of cardinaly 2

only one $\{1,2\}$



closed foam F

$F_1(c)$ - union of facets colored $\{1\}$ (contains 1)

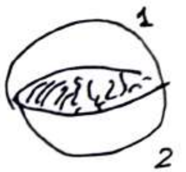
$F_2(c)$ - union of facets colored $\{2\}$ (contains 2 in the coloring)

$F_1(c), F_2(c)$ - closed surfaces in $\mathbb{R}^3$, orientable, have even Euler char

$F_{12}(c) := F_1(c) \Delta F_2(c)$ symmetric difference, union of thin facets.

$F_{12} = F_{12}(c)$ does not depend on c; closed surface in $\mathbb{R}^3$, also

Θ- foam



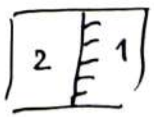
$F_1(c)$ 
$\cong$
S^2

$F_2(c)$ 
$\cong$
S^2

$F_{12} =$  $\cong S^2$

for $SL(2)$ foams has sign

$$s(S,c) = \theta_1(c) + \frac{\chi(\bar{S}_1(c))}{2}$$



circles point into 1

cap off facets colored 1 by disks

eval of $GL(2)$ foams

On to change to $\frac{\chi(F_1(c))}{2}$ everywhere

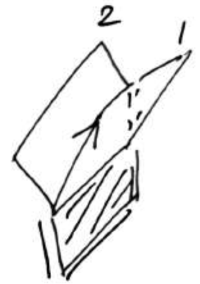
Sign $\quad s(F,c) = \theta^+(c) + \frac{\chi(F_2(c))}{2}$

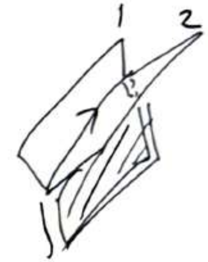
$$\langle F, c\rangle = (-1)^{s(F,c)} \frac{x_1^{d_1(c)} x_2^{d_2(c)}}{(x_1 - x_2)^{\chi(F_{12}(c))/2}}$$

does not depend on c, F_{12} – thin surface of F.

$\frac{\chi(F_2(c))}{2}$ vs $\frac{\chi(F_1(c))}{2}$

match with [BHPW]





positive $\quad$ negative

$\theta^+(c)$ $\quad$ $\theta^-(c)$

\# pos. circ $\quad$ \# neg. circles

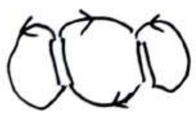
$$\langle F\rangle = \sum_c \langle F, c\rangle$$

2^k terms, k \# of components of F_{12}

2 colorings for each component, $\langle F\rangle \in R = \mathbb{Z}[E_1, E_2] = \mathbb{Z}[x_1, x_2]^{S_2}$

symm. functions.

$\Rightarrow$ get state spaces for planar $GL(2)$ MOY graphs



$\Gamma \rightsquigarrow \langle \Gamma\rangle$

universal construction

Prop $\langle \Gamma\rangle$ is a graded free ab. group of rank $(q+q^{-1})^k$

k \# of circles in $\mathrm{thin}(\Gamma)$.

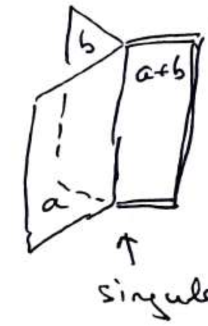
 $\leftarrow$ thin (Γ)

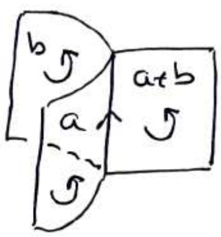
$\Rightarrow$ link homology, etc.

GL(N) foam evaluation — Robert–Wagner arXiv 2017.

GL(N) foams
$F \subset \mathbb{R}^3$

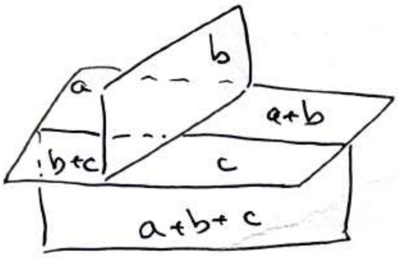
facets thickness a, $1 \le a \le N$



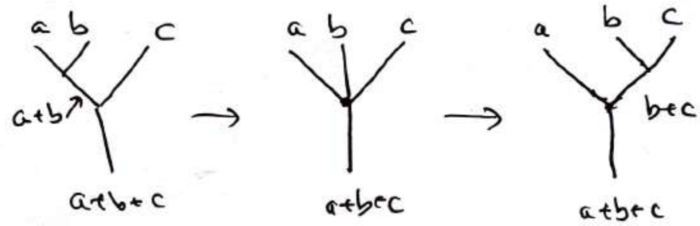


Orientation convention.
Thick facet has compatible orientation with thin facets.
Thin facets have opposite orientation

singular vertices:



cross-sections



an orientation of a single facet or a single edge determines an orientation of all ~~facets~~ facets & edges of a connected foam.

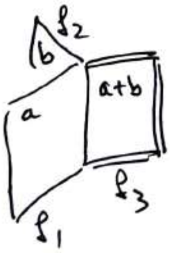
dots  an element of $Sym_a = \mathbb{Z}[y_1 .. y_a]^{S_a}$ Symm f's in a variables homogeneous (so foam has a degree)
deg $y_i = 2$

dots on thickness 1 facets - usual dots  $\rightarrow y_i^n$

$I_N = \{1, 2 .., N\}$ set of colors

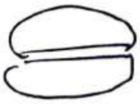
An (admissible) coloring c of F is a map $f(F) \xrightarrow{c}$ Subsets of I_N
facets

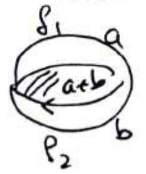
facet f of thickness $a \longrightarrow c(f) \subset I_N$ of cardinality a.



$c(f_3) = c(f_1) \sqcup c(f_2)$ $\qquad (a+b) = a+b$
disjoint union

Examples:  $c \subset I_N$, $|c| = a$





$c(f_1), c(f_2) \subset I_N$ $\quad |c(f_1)| = a$, $|c(f_2)| = b$,
disjoint

Ex Choose $1 \le i < j \le N$ pair of colors, coloring c

$F_i(c)$ - union of facets that contain color i.

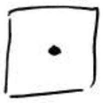
Prop 1. $F_i(c)$ is a closed orientable surface in $\mathbb{R}^3$. (exercise: check around singular points)

Let $F_{ij}(c) = F_i(c) \Delta F_j(c)$ symmetric difference, union of facets that contain one color but not both $\{i, j\}$

Prop 2 $F_{ij}(c)$ is a closed orientable surface in $\mathbb{R}^3$.

$F_i(c)$, $F_{ij}(c)$ have even Euler characteristic.

Remark: Given F, c, the union of facet that contain at least one of i, j is a $GL(2)$ foam

facet f  function $g \in \mathrm{Sym}_a(y_1 .. y_a)$

$GL(2)$ foam.
no singular vertices.
thin & thick facets.

coloring c, $c(f) = \{i_1, i_2 .., i_a\} \subset I_N$

replace $y_1 .. y_a$ by $x_{i_1} ... x_{i_a}$

$\Rightarrow g$ is now a function in $\mathbb{Z}[x_1, .. x_N] = R'$

$P_f(c) \in R'$ function associated to facet f (all dots together), coloring c

$$P(F,c) = \prod_{f \in f(F) \text{ facets}} P_f(c) \quad \leftarrow \text{Contribution from generalized dots}$$

$$Q(F,c) = \prod_{1 \le i < j \le N} (x_i - x_j)^{\chi(F_{ij}(c))/2} \quad \text{integer}$$

choose order on $\{1, 2, .. N\}$ $\quad x_i - x_j$ vs $x_j - x_i$

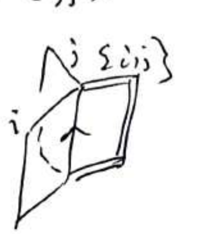
$$s(F,c) = \theta^+(c) + \sum_{i=1}^{N} i\, \chi(F_i(c))/2 \quad \text{integer}$$

$$\theta^+(c) = \sum_{i<j} \theta^+_{ij}(c)$$

$$\langle F, c \rangle = (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)} \in R'' = R'\left[\frac{1}{x_i - x_j}\right]_{i<j}$$

Look at $GL(2)$ foam for (i,j). Thin/thick facets

or



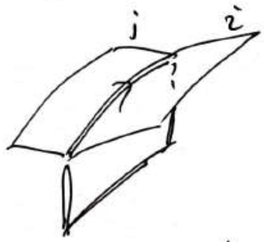
Look along singular edge, in its orientation direction

$i<j$

singular circle is

position double facet below

positive



smaller # on right

negative

smaller # on left

$\Theta^+_{ij}(c)$ = # of positive (i,j)-circles $\qquad \Theta^+(c)=\sum_{i<j}\Theta^+_{ij}(c)$

$$\sum_{i=1}^{N} i\chi(F_i(c))/2 = \chi(F_1(c))/2 + \underbrace{\frac{2\chi(F_2(c))}{2}}_{\overset{\shortparallel}{0} \bmod 2} + \frac{3\chi(F_3(c))}{2} + \dots + \frac{N\chi(F_N(c))}{2} \qquad (*)$$

$$s(F,c)=\Theta^+(c)+\sum_{i=1}^{N} i\chi(F_i(c))/2 \qquad (-1)^{s(F,c)} = (-1)^{\Theta_+(c)+\frac{\chi(F_1(c))+\chi(F_3(c))+\dots}{2}}$$

all odd indices

slightly modified eval: change to

$\chi(F_2(c))+\chi(F_4(c))+\dots$

all odd indices $\qquad (**)$

$$\langle F,c\rangle = (-1)^{s(F,c)}\frac{P(F,c)}{Q(F,c)} =$$

contributions of dots on facets

$$=(-1)^{\Theta_+(c)+\frac{\chi_1(c)+\chi_3(c)+\dots}{2}} \frac{\prod_f P_f(c)}{\prod_{i<j}(x_i-x_j)^{\chi_{ij}(c)/2}}$$

Shorthand $\chi_i(c)=\chi(F_i(c))$

$\chi_{ij}(c)=\chi(F_{ij}(c))$

$$\langle F\rangle=\sum_c \langle F,c\rangle$$

symm polyn

<u>Thm</u> (R–W) $\langle F\rangle \in R = \mathrm{Sym}_N(x_1\dots x_N)$

Do $N=2$ case. Use sign contribution (**) to match with Skein relations in [BHPW] Beliakova-Hogancamp-Putyra-Wehrli arXiv 2019 1903.12194

$N=2$ $\quad x_1, x_2 \qquad s(F,c) = \theta^+_{12}(c) + \frac{\chi_2(c)}{2} \leftarrow \chi(F_2(c))$

assume no dots on thick facets

$$\langle F, c\rangle = (-1)^{s(F,c)} \frac{x_1^{d_1(c)}\, x_2^{d_2(c)}}{(x_1-x_2)^{\chi_{12}(c)/2}}$$

$d_i(c)$ # dots on facets color i (thin facets)

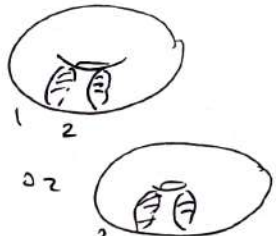
$\chi_{12}(c) = F_{12}(c)$.

$F_{12}(c)$ - does not depend on c, union of all thin facets

$F = F_{12} \cup$ double facets (thickness 2)

double facets are colored $\{1,2\}$

Each component of F_{12} has 2 colorings (checkerboard)



Examples 1) S^2  n

$\theta^+ = 0 \qquad 1$  n $\quad F_1 = S^2, \ F_2 = \emptyset \qquad s(F,c_1) = 0 + \frac{0}{2} = 0$

coloring c_1 $\qquad \langle F, c_1\rangle = \frac{x_1^n}{x_1-x_2}$

$$\langle S^2_n\rangle = \frac{x_1^n - x_2^n}{x_1-x_2} = h_{n-1}(x_1,x_2)$$

complete symm f'n

2  n $\quad c_2 \quad F_1 = \emptyset, \ F_2 = S^2 \qquad s(F,c_2) = 0 + \frac{2}{2} = 1$

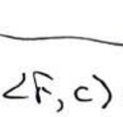
$\langle F, c_2\rangle = - \frac{x_2^n}{x_1-x_2}$

$n=0$ $\quad \langle$  $\rangle = 0 \qquad \langle$  $\rangle = 1$

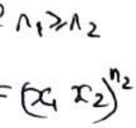
2) n $\quad T^2$ 2-torus $\qquad \langle$  $\rangle = x_1 + x_2$

c_1 coloring $\quad F_1 = T^2, \ F_2 = \emptyset \qquad s(F,c_1) = 0$

c_2 coloring $\qquad \langle T^2_n\rangle = x_1^n + x_2^n$ $\quad$ no signs, no denominators



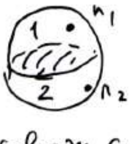
positive



negative

3) Exercise: genus g 

4) theta-foam



coloring c_1

positive circle $\theta^+ = 1$

$\chi_2(c_1) = \chi(F_2(c_1)) = \chi(S^2) = 2$

$s(F,c_1) = \theta^+ + \frac{\chi_2(c_1)}{2} = 1 + \frac{2}{2} = 2$

sign $(-1)^{s(F,c_1)} = 0$

$$\langle F, c_1\rangle = \frac{x_1^{n_1} x_2^{n_2}}{x_1-x_2}$$

coloring c_2

negative circle $\theta^+ = 0$ same $\chi_2(c_2) = \chi_2(c_1)$

$s(F,c_2) = 0 + 1 = 1 \qquad \langle F, c_2\rangle = \frac{x_1^{n_2} x_2^{n_1}}{x_1-x_2}$

$$\langle F, c\rangle = \frac{x_1^{n_1} x_2^{n_2} - x_1^{n_2} x_2^{n_1}}{x_1-x_2}$$

if $n_1 \geq n_2$

$$= (x_1 x_2)^{n_2} \frac{x_1^{n_1-n_2} - x_2^{n_2-n_2}}{x_1-x_2} =$$

$$= (x_1 x_2)^{n_2} h_{n_1-n_2-1}(x_1,x_2).$$

$$\langle F, c\rangle = (-1)^{\theta^+(c) + \sum_{i=1}^{N} (i-1)\, \chi(F_i(c))/2} \frac{P(F,c)}{Q(F,c)}$$

$$\theta^+(c) = \sum_{i<j} \theta^+_{ij}(c)$$

$$F_{i\cup j}(c) = F_i(c) \cup F_j(c)$$

↑

GL(2) foam

$$\langle F_{i\cup j}, C\rangle_{i,j} = (-1)^{\theta^+_{ij}(c) + \chi(F_j(c))/2} \frac{P(F_{i\cup j}, c)}{Q(F_{i\cup j}, c)}$$

$F_{i\cup j}(c)$

$$Q(F,c) = \prod_{i<j} (x_i - x_j)^{\chi(F_{ij}(c))/2} =$$

$$= \prod Q_{ij}(F_{i\cup j}, c)$$

$F_i(c)$

$$(-1)^{s(F,c)} = \prod_{i<j} (-1)^{s(F_{i\cup j}, c)}$$

$F_j(c)$

$$\langle F, c\rangle = \prod_{i<j} \frac{(-1)^{s(F_{i\cup j}, c)}}{(x_i - x_j)^{\chi(F_{ij}(c))/2}} P(F,c)$$

$F_{ij}(c)$

If no dots,

$$\langle F, c\rangle = \prod_{i<j} \langle F_{i\cup j}, c\rangle$$

↑

GL(2) foam evaluations.