Non-orientable knot cobordisms and torsion order in Floer homologies

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NON-ORIENTABLE KNOT COBORDISMS
AND
TORSION ORDER IN FLOER HOMOLOGIES

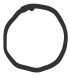
(joint work with Sherry Gong)

① DEFINITIONS and MOTIVATION

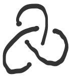
Def: KNOT = a closed, connected, smooth 1-submfd of S^3, up to isotopy

LINK = a closed, smooth 1-submfd of S^3, up to isotopy

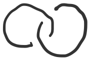
e.g.



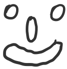
unKnot U



trefoil

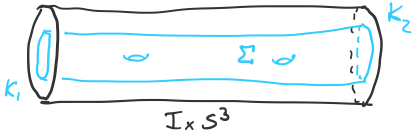


Hopf link



4-cpt unlink

Def: A KNOT COBORDISM from K_1 to K_2 is a smooth, prop. emb. $\Sigma^2 \subseteq I \times S^3$ w/ $\partial\Sigma = \{0\} \times K_1 \sqcup \{1\} \times K_2$



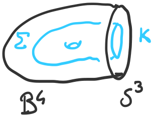
⊛ orientable if Σ orient. → GENUS $= g(\Sigma)$

⊛ non-orient. if Σ non-orient. → (non-orientable) GENUS
$$\gamma(\Sigma) = -\chi(\Sigma)$$

⊛ concordance if Σ is an annulus

Assumption: today all surfaces will be connected.

Special case: if $K_1 = U$
$K_2 = K$



Classical question: Given K, what restriction do I get on Σ?

Def: (OR.) SLICE GENUS:

$$g_4(K) = \min\left\{ g(\Sigma) \;\middle|\; \begin{array}{l} \Sigma \subseteq B^4 \text{ smooth, orientable} \\ \partial\Sigma = K \end{array} \right\}$$

Def NON-OR. SLICE GENUS

$$\gamma_4(K) = \min\left\{ \gamma(\Sigma) \;\middle|\; \begin{array}{l} \Sigma \subseteq B^4 \text{ smooth, } \overset{\text{crosscapbody}}{\text{non}\cancel{\text{-orient.}}} \\ \partial\Sigma = K \end{array} \right\}$$

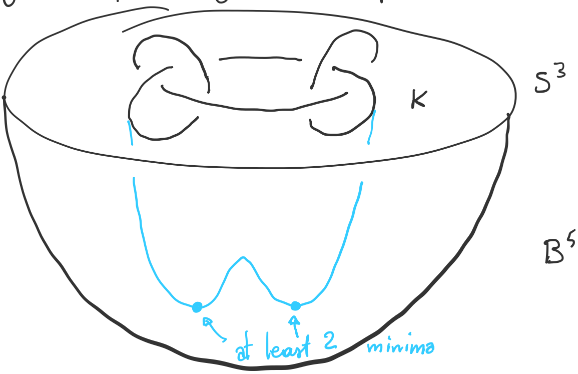
Def: CROSSCAPBODY = non-orient surface or genus-0 or. surface.

Some results:

* Classical: $g_4(K)$ can be arbitrarily large

* Batson '12: $\gamma_4(K)$ can be arbitrarily large

Q: Given Σ of minimal genus, how complicated must the embedding be?



Spoiler: Juhász-Miller-Zemke for orientable cobordisms
Gong-M. for unorientable cobordisms.

② TORSION ORDER IN FLOER HOMOLOGIES

⊛ Knot Floer homology

$K \subseteq S^3 \xrightarrow{\qquad} HFK^-(K)$ $\quad$ $\mathbb{F}[U]$-module

(Ozsváth-Szabó, Rasmussen)

$$HFK^-(K) \cong \mathbb{F}[U] \oplus \underbrace{\frac{\mathbb{F}[U]}{(U^{n_1})} \oplus \dots \oplus \frac{\mathbb{F}[U]}{(U^{n_k})}}_{\text{Tors}}$$

Def: $\mathrm{Ord}_{HFK^-}(K) = \min\{n \mid U^n \cdot \mathrm{Tors} = 0\}$

⊛ Unoriented Knot Floer homology

$K \subseteq S^3 \xrightarrow{\qquad} HFK'(K)$ $\quad$ $\mathbb{F}[U]$-module

(Ozsváth-Stipsicz-Szabó)

Def: $\mathrm{Ord}_U(K) = \min\{ n \mid U^n \cdot \mathrm{Tors}(HFK'(K)) \}$

⊛ Instanton Floer homology

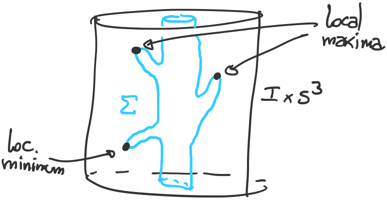
$K \subseteq S^3 \xrightarrow[\text{(Kronheimer-Mrowka)}]{} I^{\#}(K)$ module over a comm. ring S w/ special elements P & Q

Def: $\mathrm{Ord}_I(K) = \min\{ n \mid P^n \cdot \mathrm{Tors}(I^{\#}(K)) = 0 \}$

Rem: $\Sigma \hookrightarrow I \times S^3 \longrightarrow I$

height function

no local minima/maxima



Thm (Juhász-Miller-Zemke '19)
Let Σ be an orient. cob. from K_1 to K_2 w/ M local maxima. Then:

$$\mathrm{Ord}_{HFK^-}(K_1) \leq \max\{M, \mathrm{Ord}_{HFK^-}(K_2)\} + 2 \cdot g(\Sigma)$$

Thm (Geng-M.)
Let Σ be a non-orient. cob. from K_1 to K_2 w/ M local maxima. Then:

$$\mathrm{Ord}_U(K_1) \leq \max\{M, \mathrm{Ord}_U(K_2)\} + \gamma(\Sigma) + 1$$

$$\mathrm{Ord}_I(K_1) \leq \max\{M, \mathrm{Ord}_I(K_2)\} + \gamma(\Sigma)$$

③ APPLICATIONS

⊛ Unorientable ribbon cobordisms

Def: A cobordism Σ is called RIBBON, if it has no local maxima.

Cor: If $\exists$ unorient. ribbon cob from K_1 to K_2, then

$$Ord_U(K_1) \leq Ord_U(K_2) + \gamma(\Sigma) + 1$$

$$Ord_I(K_1) \leq Ord_I(K_2) + \gamma(\Sigma)$$

⊛ Unorientable refined cobordism distance

Def: ORIENTABLE COB. DISTANCE

$$d_o(K_1, K_2) = \min\{g(\Sigma) \mid \Sigma \text{ or. cob. } K_1 \to K_2\} = g_4(K_1 \# \overline{K_2})$$

Def: UNORIENTABLE COB. DISTANCE

$$d_u(K_1, K_2) = \min\{\gamma(\Sigma) \mid \Sigma \text{ crosscapbody } K_1 \to K_2\} = \gamma_4(K_1 \# \overline{K_2})$$

Both are distances on $\{\text{Knots}\}/\text{concordance}$.

Refined distances

Def: Let $\Sigma \subseteq I \times S^3$ a knot cobordism w/ m minima & M maxima.

$$|\Sigma| = \max\{m, M\} - \chi(\Sigma)$$

Def: $d_o^r(K_1, K_2) = \min\{|\Sigma| \mid \Sigma \text{ or. cob. } K_1 \to K_2\}$

$d_u^r(K_1, K_2) = \min\{|\Sigma| \mid \Sigma \text{ crosscapbody } K_1 \to K_2\}$

Rem: $d_*^r(K_1, K_2) = 0 \Rightarrow |\Sigma| = 0 \Rightarrow K_1 = K_2$

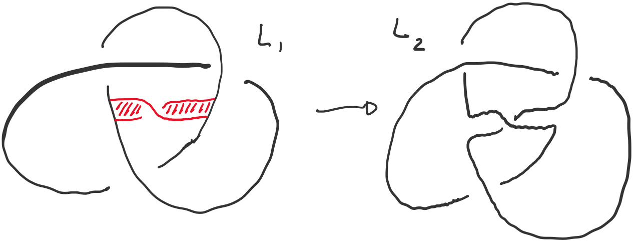
Distances on $\{\text{Knots}\}$.

Cor: $|Ord_I(K_1) - Ord_I(K_2)| \leq d_u^r(K_1, K_2)$

$|Ord_u(K_1) - Ord_u(K_2)| \leq d_u^r(K_1, K_2) + 1$

⊛ Unoriented band-unlinking number

Def: BAND SURGERY



Rem: this gives a $\chi = -1$ cobordism Σ from L_1 to L_2

Two cases: ORIENTABLE if Σ or. $(|L_1| \neq |L_2|)$
UNORIENTABLE if Σ unor. $(|L_1| = |L_2|)$

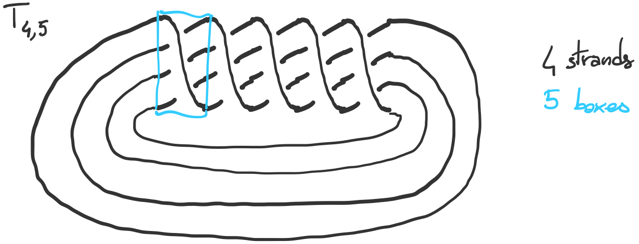
Def: $bul(K)$ = min # of B.S.'s that turn K into an n-cpt unlink
$bul_o(K)$ = or.
$bul_u(K)$ = non-or.

Cor: $|Ord_I(K)| \leq bul(K)$
$|Ord_U(K)| \leq bul(K) + 1$

Rem: C.-M. Mike Wong: $|Ord_U(K)| \leq bul(K)$

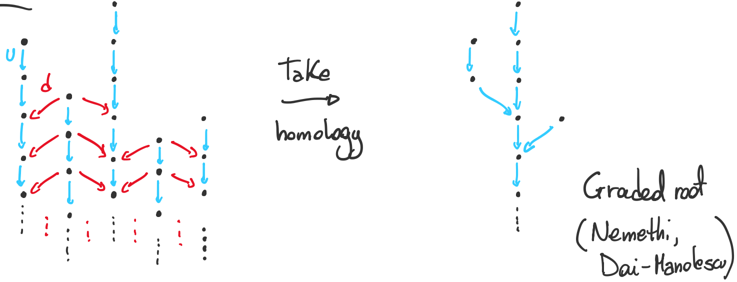
④ EXAMPLE

Def: TORUS KNOT $T_{p,q} = \left\{ (w,z) \in \mathbb{C}^2 \,\middle|\, w^p + z^q = 0 \right\} \cap S^3_\varepsilon$



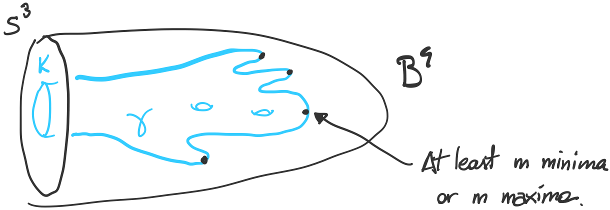
Lemma: $\mathrm{Ord}_U(T_{n,n+1}) = \lfloor n/2 \rfloor$

Idea:



Cor: For all $\gamma \geq 1$, $m \geq 1$ ∃ a Knot $K_{\gamma,m}$ w/

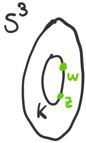
- $d_u(K_{\gamma,m}, U) = \gamma$
- $d_u^r(K_{\gamma,m}, U) \geq \gamma + m$



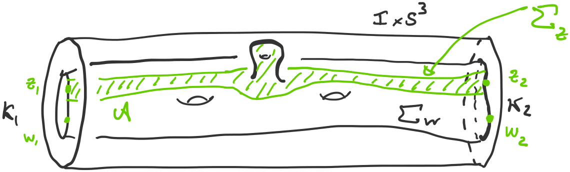
INTERLUDE

SOME TECHNICAL DETAILS

① KNOT FLOER HOMOLOGY



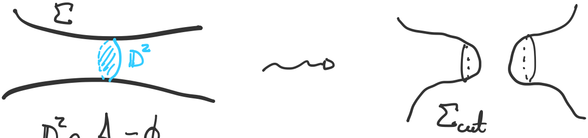
$$\rightsquigarrow HFK^-(K,w,z)$$



$$\mathcal{W} = (\Sigma, \mathcal{A})$$

$$\begin{tikzcd} {} \arrow[d] & {} \arrow[d, squiggly] & {} \arrow[d] \\ \mathcal{F}_W : HFK^-(K_1,w_1,z_1) \arrow[rr, "{[\text{Zemke}]}"'] & {} & HFK^-(K_2,w_2,z_2) \end{tikzcd}$$ U-equiv.

Def: COMPRESSION on $(\Sigma, \mathcal{A})$

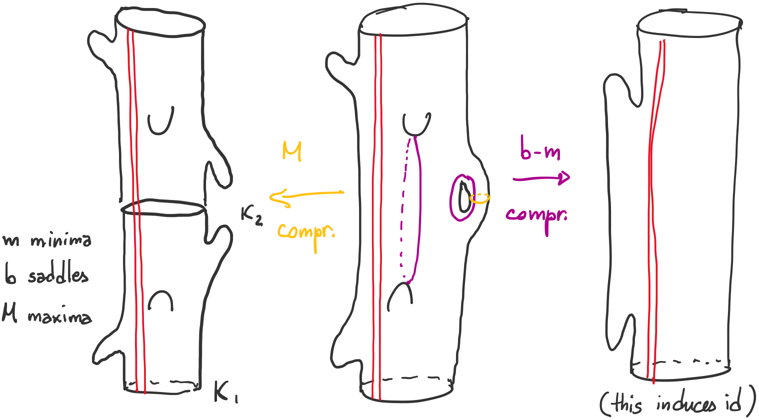


$D^2 \cap \mathcal{A} = \phi$

$\mathcal{W} \longrightarrow \mathcal{W}_{cut}$

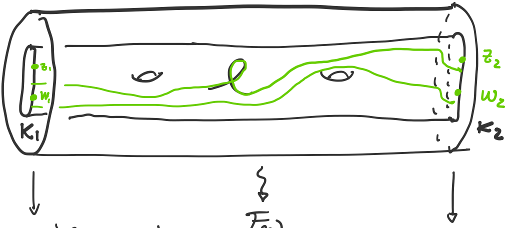
Lemma (Zemke): $\mathcal{F}_{\mathcal{W}_{cut}} = U \cdot \mathcal{F}_{\mathcal{W}}$.

Outline of Juhász-Miller-Zemke



$$\Longrightarrow \quad U^M \cdot F_{\bar{\Sigma} \circ \Sigma} = U^{b-m} \cdot id$$

② UNORIENTED KNOT FLOER HOMOLOGY

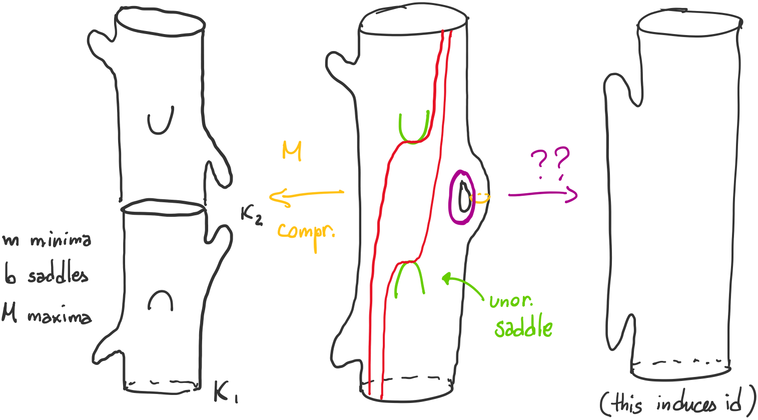


Σ_w and Σ_z orientable.

$\mathcal{W}=(\Sigma,\mathcal{A})$

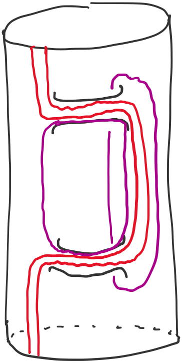
$$HFK'(K_1,w_1,z_1) \xrightarrow[{[F_{an}]}]{F_{\mathcal{W}}} HFK'(K_2,w_2,z_2)$$

Pf:



Step 1: replace unorientable tube w/ orientable tube

Step 2: Add extra handle:



⤳ compress and win

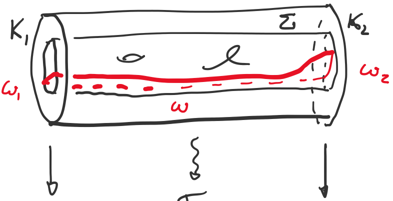
(but you get an extra +1)

③ INSTANTON FLOER HOMOLOGY

$$\left.\begin{array}{l} L \subseteq S^3 \ \text{link} \\ \theta \subseteq S^3 \setminus L \ \text{std } \theta \text{ graph} \\ \omega = 1\text{-submfd of } S^3 \text{ w/ } \partial\omega \subseteq L \cup \theta \end{array}\right\} \longrightarrow I^\#(L)_\omega$$

$I^\#(L)_\omega$ is a module over a comm. ring S with special elt's P & Q.

Note: $\omega = \phi$ is allowed.



ω = 2-submfd w/ corners

$\mathcal{W} = (\Sigma, \omega)$

$$\begin{tikzcd} I^\#(K_1)_{\omega_1} \arrow[r, "F_{\mathcal{W}}", "{[K\text{-}M]}"'] & I^\#(K_2)_{\omega_2} \end{tikzcd}$$

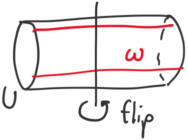
Properties:

⊛ F_W depends only on $[\partial\omega] \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$

⊛ neck cutting (if $\partial\omega = \emptyset$ and Σ_{cut} connected)

$$F_{W_{cut}} = P \cdot F_W$$

Pf: Step 1:



⇝ $F_W = id$

⇝ replace flip w/ oriented tube

Step 2: Find ω on Σ such that

⊛ ω is as above in the flipping tube

⊛ after replacing w/ oriented tube, $[\partial\omega] = 0$

⇝ get rid of that and use neck cutting. □