The diffeomorphism group of a 4-manifold

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Background

Basic object: X a smooth oriented manifold, $\text{Diff}(X)$ its group of orientation-preserving diffeomorphisms.

$\text{Diff}(X)$ is large (infinite dimensional) and complicated.

Study its algebraic topology. Simplest: set of path components.

Definition. Diffeomorphisms f_0 and f_1 are *isotopic* ($f_0 \sim f_1$) if they are connected by a path of diffeomorphisms.

Isotopy classes are a group $\pi_0(\text{Diff}(X))$.

Similarly: topological isotopy ($f_0 \sim_{top} f_1$) of homeomorphisms.

Background

Comparison: Natural map $\mathrm{Diff}(X) \to \mathrm{Homeo}(X)$ induces

$$\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Homeo}(X)).$$

Is it injective? Surjective? What about higher homotopy groups?

Classic result: (Milnor 1958). Exotic 7-sphere; gives diffeomorphism $f : S^6 \to S^6$ topologically isotopic to $\mathbf{1}_{S^6}$ but not smoothly isotopic to $\mathbf{1}_{S^6}$.

From now on, concentrate on dimension four.

Will find new phenomena compared with higher dimensions.

Isotopy and TOP isotopy

Topological isotopy is well-understood in dimension 4 when $\pi_1(X) = \{1\}$:

$$f \sim_{top} \mathbf{1}_X$$

Quinn; Perron: $f \simeq \mathbf{1}_X \Longleftrightarrow f \sim_{top} \mathbf{1}_X$

$f \sim_{top} \mathbf{1}_X \Longleftrightarrow f_* = \mathrm{id}$ on $H_2(X)$

$$f \simeq \mathbf{1}_X \Longleftrightarrow f_* = \mathrm{id} \text{ on } H_2(X)$$

Cochran-Habeggar

The smooth case is much more complicated.

Theorem 1. (R. 1998) *There are 4-manifolds Z_r for which*

$$\ker\left[\pi_0(\mathrm{Diff}(Z_r)) \to \pi_0(\mathrm{Homeo}(Z_r))\right]$$

contains a $\mathbb{Z}^r$ summand.

Stabilization

Theorem 1 depends on a stabilization property of 4-manifolds:

There are non-diffeomorphic 4-manifolds X_1 and X_2 such that

$$X_1 \# S^2 \times S^2 \cong X_2 \# S^2 \times S^2.$$

Auckly (1998) gave an explicit infinite family of manifolds $\{X_k\}$ with this property.

Idea: More stabilizations $\Rightarrow$ elements in higher homotopy groups.

Theorem 2 (Auckly-R. 2020) *There are 4-manifolds $Z_{k,r}$ for which*

$$\ker\left[\pi_k(\mathrm{Diff}(Z_{k,r})) \to \pi_k(\mathrm{Homeo}(Z_{k,r}))\right]$$

contains a $\mathbb{Z}^r$ summand.

Ordinary gauge theory

Brief review of Donaldson theory: Degree 0 invariant.

Suppose X has $b^2_+(X)$ odd and > 1, and $P \to X$ is an SO(3) bundle with

$$d(P) = 2p_1(P) - 3(b^2_+(X) + 1) = 0.$$

For generic choice of Riemannian metric g on X, the *ASD moduli space*

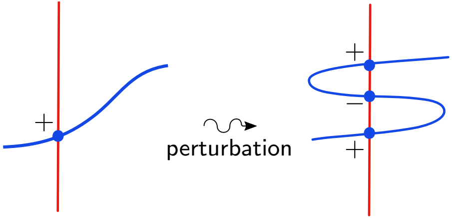
$$\mathcal{M}(P; g) = \left\{ \text{Connections A on P with } F_A^+ = 0 \right\} / \, \text{Aut}(P)$$

is a smooth *compact, oriented* 0-manifold.

Definition: $D(X; P) = \#\mathcal{M}(P; g)$; it's a diffeomorphism invariant.

Ordinary gauge theory

Analogy 1: Degree 0 invariant is like an intersection number–it counts (with signs) solutions to an equation.



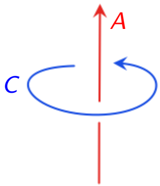
- Expected dimension of intersection (like $d(P)$) is 0.
- Count with signs doesn't change under deformation.
- More accurately: Count of solutions to (nonlinear) elliptic equation with index 0; the signed count doesn't change under metric deformation.

Invariants of families

We detect interesting maps of S^k to $\mathrm{Diff}(X)$ using *parameterized (or family) gauge theory*.

Analogy 2: A family invariant is like a linking number.

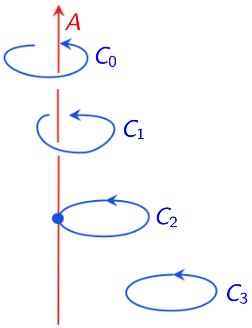
Imagine manifolds A and C in $\mathbb{R}^m$ with $\dim(A) + \dim(C) < m$.



Expected dimension of intersection $(\dim(A) + \dim(C)) - m < 0$ so expect them to be disjoint.

Invariants of families

Let C move in a $m - (\dim(A) + \dim(C))$ parameter family $\{C_t\}$ and count the intersections.



Signed count is independent of how you get from C_0 to C_3.

Gauge theory for families

Suppose that $P \to X$ is such that $d(P) = -n < 0$; then $\mathcal{M}(P, g)$ is empty for generic g.

We get an invariant from family of manifolds X_b and Riemannian metrics g_b for $b \in B$, where $\dim(B) = n$.

Definition: *For an* SO(3) *bundle* $\mathcal{P} \to B \times X$ *define the parameterized moduli space*

$$\mathcal{M}(\mathcal{P}, \{g\}) = \bigcup_{b \in B} \{b\} \times \mathcal{M}(P_b, g_b).$$

Then generically $\mathcal{M}(\mathcal{P}, \{g\})$ is 0-dimensional and we can again count points with sign.

Gauge theory for families

Idea goes back to Donaldson (1989); developed by R. (1998), Li-Liu (2001) for Seiberg-Witten equations

Recent work: Baraglia, Konno, Kronheimer-Mrowka, J. Lin ...

Resulting (signed) count is denoted $D(\mathcal{P}, \{g_b\})$.

If $b^2_+(X) > \dim(B) + 1$ then it's independent of $\{g_b\}$ and is denoted $D(\mathcal{P})$.

These results use the fact that $\mathcal{M}et(X)$, the space of Riemannian metrics, is contractible.

Families of diffeomorphisms

Suppose $\alpha : S^k \to \mathrm{Diff}(X)$ and that g is a metric on X. Then α defines a map $S^k \to \mathcal{M}et(X)$ by $g \to \alpha^* g$.

This extends to a map $A : B^{k+1} \to \mathcal{M}et(X)$ that gives a family $\{A(z) \mid z \in B^{k+1}\}$ of metrics.

Definition: *Let $P \to X$ be an* SO(3) *bundle with $d(P) = -(k+1)$. The count of points in the parameterized moduli space*

$$\bigcup_{z \in B^{k+1}} \mathcal{M}(P, A(z))$$

defines an integer $D^P(\alpha)$.

Properties of $D^P(\alpha)$

Proposition: *Suppose* $b^2_+(X) > k+2$. *Then*

1. $D^P(\alpha)$ *is independent of initial metric* g *and extension* A.
2. $D^P(\alpha)$ *depends only on the homotopy class of* $\alpha \in \pi_k(\mathrm{Diff}(X))$.
3. $D^P(\alpha)$ *is a homomorphism* $\pi_k(\mathrm{Diff}(X)) \to \mathbb{Z}$.

In proving Theorem 2, we vary P *to get a homomorphism* $\pi_k(\mathrm{Diff}(X)) \to \mathbb{Z}^r$.

For today: just one P so we write $D(\alpha)$.

Let's find some interesting families of diffeomorphisms.

Break

Spheres and diffeomorphisms

Heart of the construction: complex conjugation $C : \mathbb{C}P^2 \to \mathbb{C}P^2$.

A sphere $S \subset M^4$ with $S \cdot S = \pm 1$ gives a decomposition

$$M = M' \# \pm\mathbb{C}P^2 \text{ and diffeomorphism } \rho^S : M \to M$$

defined by $\rho^S = \mathbf{1}_{M'} \# C$.

The pair (M, S) determines M' (*blowing down*: $M' = M/S$).

Idea: ρ^S also remembers M': $D(\rho^S)$ determines $D(M')$.

Stabilizing manifolds

Fact: There are manifolds X_j with $D(X_j) \neq D(X_k)$ for $j \neq k$ but

$$X_j \# \mathbb{C}P^2 \cong X_k \# \mathbb{C}P^2.$$

Sum with $\mathbb{C}P^2$ or $S^2 \times S^2$ is a *stabilization*.

Now $X_j \# \mathbb{C}P^2$ contains a sphere S_j with $S_j \cdot S_j = 1$; can assume $S_j \simeq S_k$.

The ρ^{S_i} are interesting; we need something a bit more elaborate.

Define:

$$Z = X_j \# \underbrace{\mathbb{C}P^2 \# -\mathbb{C}P^2 \# -\mathbb{C}P^2}_{N}$$

The two copies of $-\mathbb{C}P^2$ contain spheres $E_1,\ E_2$ with self-intersection -1.

Stabilizing manifolds

Note $(S + E_1 \pm E_2)^2 = -1$ so we get a diffeomorphism on N

$$f^N = \rho^{S+E_1+E_2} \circ \rho^{S+E_1-E_2}$$

Each decomposition $Z = X_j \# N$ gives a diffeomorphism $\alpha_j = \mathbf{1}_{X_j} \# f^N$.

Lemma. *For all j and k, $\alpha_j \simeq \alpha_k$ (hence they are top isotopic).*

Theorem: (R. 1998) $D(\alpha_j) = 4D(X_j)$. So if $j \neq k$,

$$\alpha_j \circ \alpha_k^{-1} \in \ker \left[\pi_0(\mathrm{Diff}(Z)) \to \pi_0(\mathrm{Homeo}(Z))\right].$$

Stabilizing spheres and diffeomorphisms

In 1998, Dave and I made a guess: α_j and α_k are isotopic after another stabilization. It took a while ....

Theorem: (Auckly-Kim-Melvin-R. 2015) *The spheres S_j and S_k are isotopic in $Z \# \mathbb{C}P^2$ (and in $Z \# S^2 \times S^2$ or even $Z \# N$). Hence*

$$\alpha_j \# \mathbf{1}_{S^2 \times S^2} \sim \alpha_k \# \mathbf{1}_{S^2 \times S^2}.$$

Remark: This is a general phenomenon (AKMR+Schwartz 2019).

Executive summary of the rest of the talk: The stable isotopy from the above theorem gives rise to a $\mathbb{Z}$ subgroup of

$$\ker\left[\pi_1(\mathrm{Diff}(Z \# N)) \to \pi_1(\mathrm{Homeo}(Z \# N))\right].$$

Iterating this process gives higher homotopy groups.

From isotopy to loops

Fix $j \neq k$ and write $\alpha = \alpha_j \circ \alpha_k^{-1}$, and let F_t be an isotopy with

$$F_0 = \alpha \# \mathbf{1}_N \text{ and } F_1 = \mathbf{1}_{Z \# N}.$$

The isotopy F_t gives a loop β of diffeomorphisms on $Z \# N$:

$$\beta_t = F_t \circ (\alpha \# \mathbf{1}_N) \circ F_t^{-1} \circ (\alpha \# \mathbf{1}_N)^{-1}.$$

Theorem: (Auckly-R. 2019) *The loop β satisfies*

1. $\beta \in \ker [\pi_1(\mathrm{Diff}(Z \# N)) \to \pi_1(\mathrm{Homeo}(Z \# N))]$
2. $D(\beta) = 4D(\alpha) = 16(D(X_j) - D(X_k))$.

Hence β generates an infinite cyclic subgroup of that kernel.

Part 1 is relatively easy, using a topological isotopy of α to the identity.

Part 2 is a generalization of Donaldson's connected sum theorem.

Higher homotopy groups

To find elements in $\ker [\pi_k(\mathrm{Diff}) \to \pi_k(\mathrm{Homeo})]$ we iterate this process.

Connect sum again with N; then β becomes null homotopic. Use a commutator construction as above to turn this 2-parameter family of diffeomorphisms into a map of a sphere to the diffeomorphism group.

Proof of topological triviality and calculation of D invariants are similar.

About interesting families

All happy families are alike; each unhappy family is unhappy in its own way.—Leo Tolstoy, Anna Karenina.