The trace embedding lemma & spinelessness

Joint with Kyle Hayden

<u>Motivating Q:</u> given $\begin{smallmatrix}\text{SM}\\\text{TOP}\\\text{PL}\end{smallmatrix}$ manifold X^n homotopy equivalent to $\begin{smallmatrix}\text{SM}\\\text{TOP}\\\text{PL}\end{smallmatrix}$ manifold W^m with $m<n$, when does there exist an $\begin{smallmatrix}\text{SM}\\\text{TOP}\\\text{PL}\end{smallmatrix}$ embedding

$\phi : W \hookrightarrow X$ giving homotopy equivalence?

Such W is $\begin{smallmatrix}\text{SM}\\\text{TOP}\\\text{PL}\end{smallmatrix}$ <u>spine</u>, if no ϕ (but $X^n \simeq W^m$) X is $\begin{smallmatrix}\text{SM}\\\text{TOP}\\\text{PL}\end{smallmatrix}$ <u>spineless</u>

<u>today</u>: PL

$n-m=1$, Always

$n-m>2$, Always (Browder, Casson, Haefliger, Sullivan, Wall, Late '60s)

$n-m=2$, if $\pi_1(W)=1$, $n\geq 5$, always

if $\pi_1(W)\neq 1$, n odd, always

n even, spineless X^n exist

((Cappell-Shaneson '77)

Remark: when $n=4$ $m=2$ SM: for any W, spineless X exist
TOP: for any $W \not\cong S^2$, spineless X exist

reasonably easy to prove

Super common question in 4-manifolds : given $\alpha \in H_2(X^4, \mathbb{Z})$, $K \subseteq \partial X$, determine $\min \left\{ \text{genus}(\Sigma^2) \;\middle|\; \begin{array}{l} \Sigma \xhookrightarrow{\text{SM/TOP}} X \\ [\Sigma] = \alpha \\ \partial\Sigma = K \end{array} \right\}$

SM
TOP

$\Longrightarrow$ tools for obstructing $\genfrac{}{}{0pt}{}{\text{SM}}{\text{TOP}}$ spines (surfaces) abound

Recall: for $n = \begin{cases} 2 \\ 4 \end{cases}$, PL = SM, $W^2_{PL} \xhookrightarrow{PL} X_{PL} \iff W^2_{sm} \xhookrightarrow{\phi} X^4_{sm}$, ϕ sm away from finitely many cone points

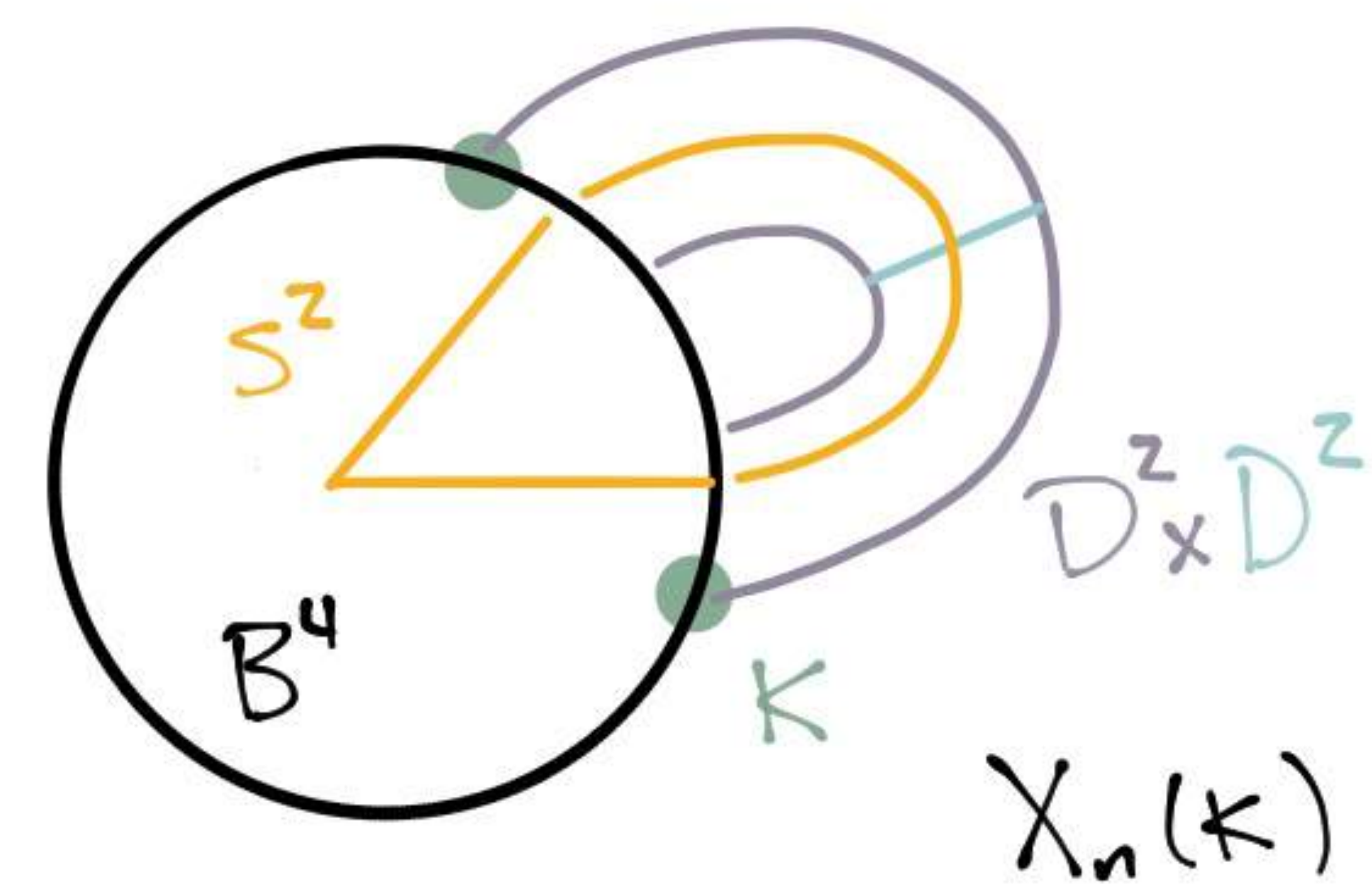
from smooth perspective, these are not that natural.

homotopy equivalent

Thm (Matsumoto '75, Levine-Lidman '18): $\forall g \in \mathbb{N}$, $\exists\, X^4 \simeq \Sigma_g^2$ but X PL spineless

g>0   g=0, Kirby 4.25

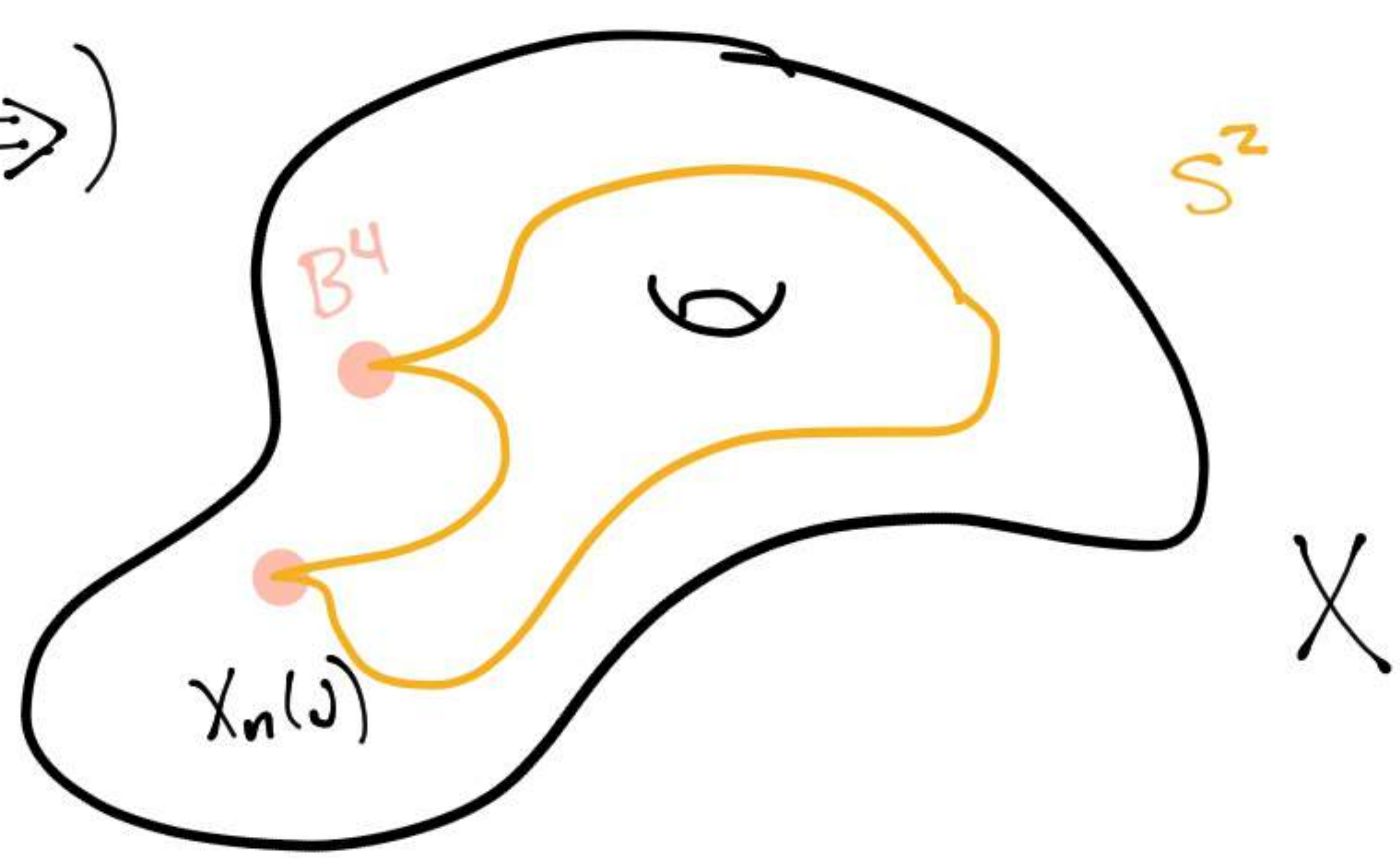
Defn: A <u>knot trace</u> $X_n(K) := B^4 \cup \left\{ \begin{matrix} n\text{-framed 2-handle} \\ \text{attached along } K \end{matrix} \right\}$



Obs:
- $\forall K, n$ $X_n(K)$ is homotopy equivalent to S^2
- $\forall K, n$ $X_n(K)$ has a PL spine
- $\forall n$, if <u>K bounds $D^2 \xhookrightarrow[sm]{} B^4$</u> then $X_n(K)$ has sm spine ← K <u>slice</u>
- $\partial X_n(K) \cong S^3_n(K)$

Lemma 1: $X \simeq S^2$ has a PL spine $\iff \exists$ some K, n s.t. $X_n(K) \xhookrightarrow[sm]{\phi} X$ w/ ϕ giving a homotopy equivalence and $X \setminus \phi(\mathring{X}_n(K))$ a homology cobordism.

sketch: ($\Longrightarrow$)



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Prop: if $Y^3 \simeq S^3_n(K)$ for some $K, |n|>1$, then $\{d(Y, \mathfrak{s})\}$ takes certain form

homology cobordant

$\mathbb{Q}$-valued invts from Heegaard Floer

homotopy equivalent

Cor:
~~Thm~~ (Levine-Lidman '18): $\exists$ $X^4 \simeq S^2$ s.t. no $\varphi: \Sigma_g \xhookrightarrow[PL]{} X$ s.t. φ gives homotopy equivalence

Sketch: build X^4 homotopy equiv to S^2 w/ $\{d(\partial X, \mathfrak{s})\}$ not of this form //

Rmk: also shows ∂X (a $\mathbb{Q}H_* S^3$) not dehn surgery on any knot in S^3.

Open: $\exists$ Y^3 not homeo to $S^3_f(L)$ for any 2-component link L?

Thm (Sivek-Zentner '19): gauge theory free pf that $\exists$ $\mathbb{Q}H_* S^3$ not homeo to $S^3_f(K)$ for some knot K

Thm (L-L): $\exists$ Y^3 which bound htpy S^2s s.t. every htpy S^2 filling is spineless

Concerns:
- spineless should depend on 4-mfld
- dont want to compute d invariants/sets of invariants

Theorem (Hayden-P.) $\forall g \in \mathbb{N}$, $n \in \mathbb{Z}$ $\exists$ $W^4 \simeq \Sigma_g^2$, $Q_W = [n]$ s.t. W is P.L. spineless and s.t $W \cong_{TOP} W'$ which has a smooth spine

Motivating Question: Find small closed exotic X^4; i.e. X^4 s.t $\exists$ W^4 w/ $W \cong_{TOP} X$, $W \ncong_{sm} X$

When working w/ ∂, want:
- obstruction not to rely too much on ∂
- sm structures "really different".

Cor: when $g = n = 0$, we produce homeomorphic manifolds W, W' homotopy equivalent to S^2 such that:

W
- is P.L. spineless
- is not diffeo to any $X_n(J)$
- admits a Stein structure
-

but W'
- has a smooth spine
- is $X_0(K)$ for some slice K
- is not a strong symplectic filling of $\partial W'$
-

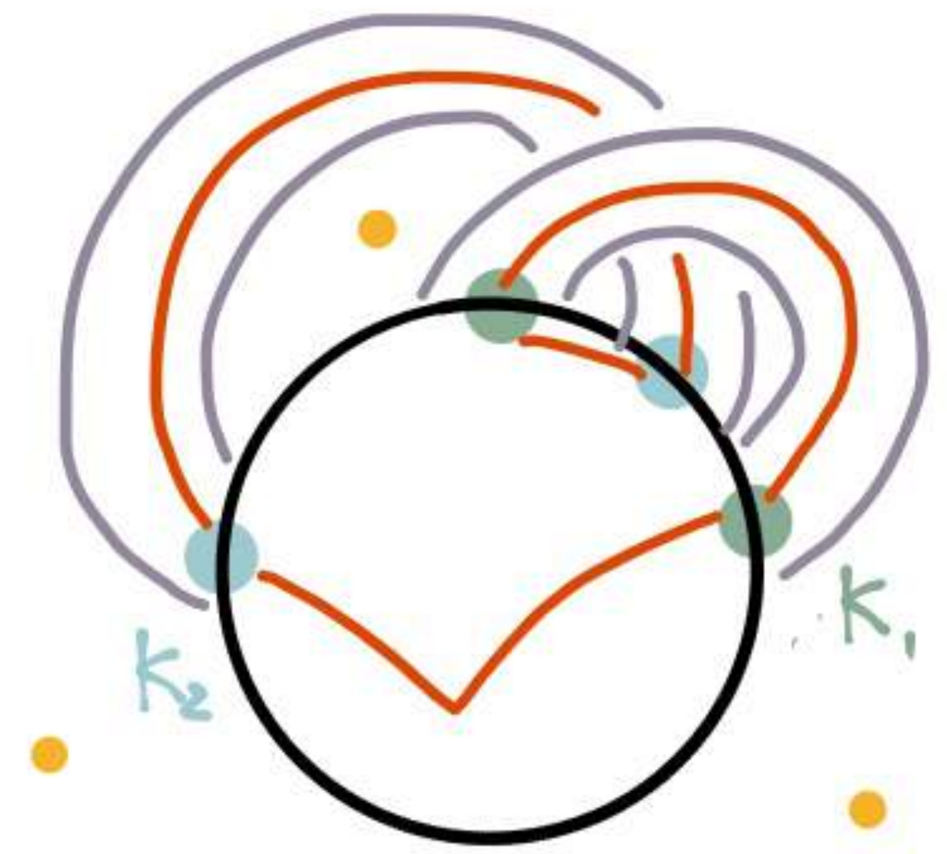
Def^n: X^4 is geometrically simply connected (g.s.c) if X admits a handle decomp w/ no 1-handles

Obs: g.s.c $\Longrightarrow$ simply connected

Open Problem (Kirby 4.18): For X^4 ~~closed~~ does s.c. $\Longrightarrow$ g.s.c?

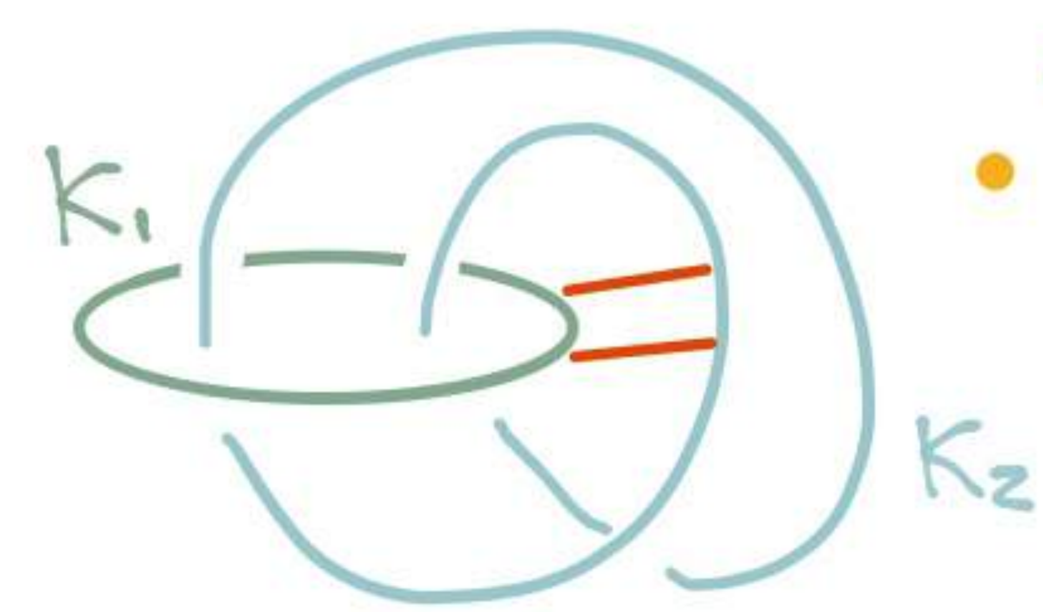
Thm (Casson '70s, Gordon '81): If X^4 contractible, $\pi_1(\partial X) \neq 1$ then X is not g.s.c.

Concerns[2]: • too much reliance on ∂
• g.s.c. should depend on smooth structure?



Cor (L-L): Any htpy S^2 Filling of Y^3 from theorem is not g.s.c.

sketch:
If X g.s.c. then every $\alpha \in H_2(X)$ can be represented by $S^2 \underset{PL}{\hookrightarrow} X$.



Thus g.s.c. $\Longrightarrow$ not spineless

Theorem (Hayden–P.) $\forall g \in \mathbb{N},\ n \in \mathbb{Z}\ \exists\ W^4 \simeq \Sigma_g^2,\ Q_W = [n]$ s.t. W is spineless and s.t $W \cong_{TOP} W'$ which has a (smooth) spine

Motivating Question: Find small closed exotic X^4; i.e. X^4 s.t $\exists\ W^4$ w/ $W \cong_{TOP} X$, $W \not\cong_{sm} X$

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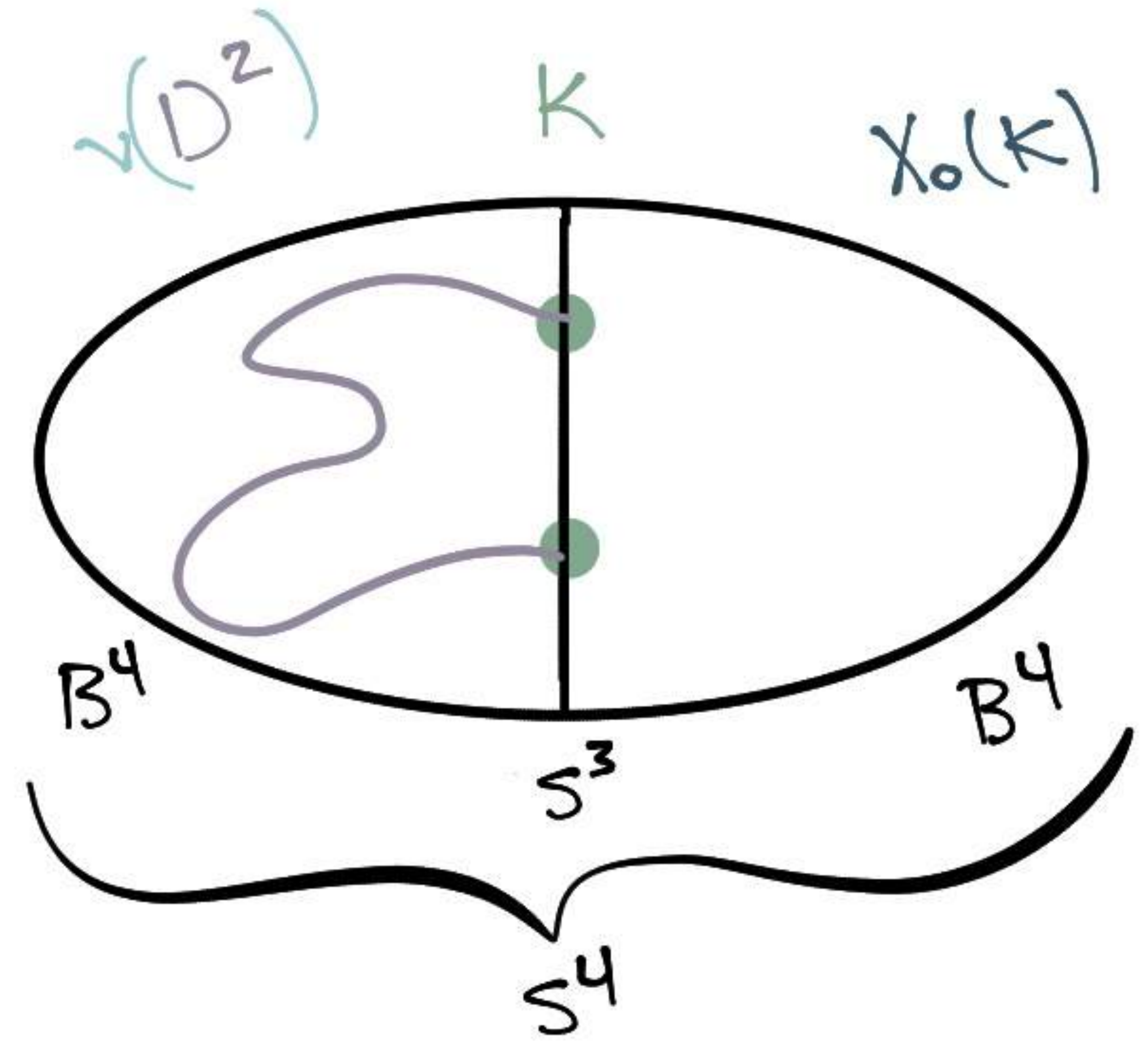
W
- is P.L. spineless
- is not diffeo to any $X_n(J)$
- admits a Stein structure
- is not g.s.c.

but W'
- has a smooth spine
- is $X_0(K)$ for some slice K
- is not a strong symplectic filling of $\partial W'$
- is g.s.c.

Lemma 2: $X_0(K) \xhookrightarrow[sm]{} S^4 \Longleftarrow K \subseteq S^3 = \partial B^4$ bounds $D^2 \xhookrightarrow[sm]{} B^4$ (K is slice)

sketch: $(\Longleftarrow)$



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Theorem (Hayden-P.) $\forall g \in \mathbb{N}, n \in \mathbb{Z}\ \exists\ W^4 \simeq \Sigma^2_g$, $Q_W = [n]$ s.t. W is spineless and s.t $W \cong_{TOP} W'$ which has a (smooth) spine

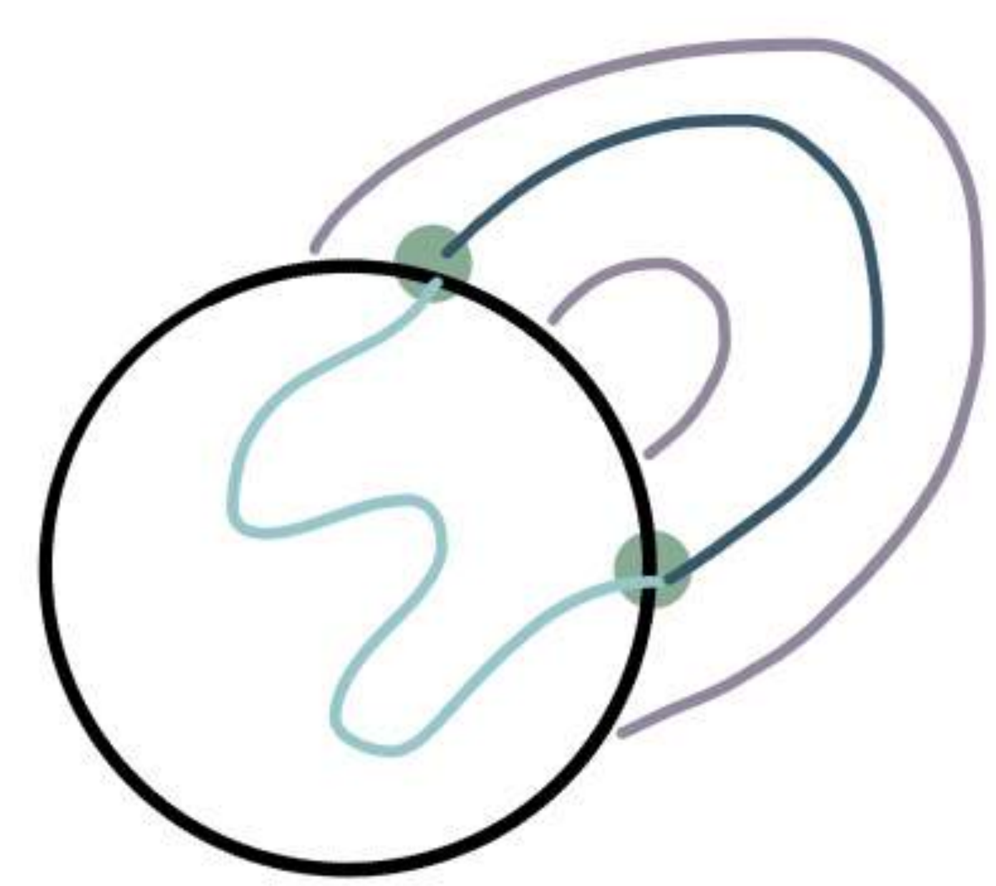
pf sketch: Build W s.t. 0) $Q_W = [0]$, $W \simeq S^2$ homotopy equivalent
($n=g=0$)
1) W has no smooth spine
2) $W \underset{sm}{\hookrightarrow} S^4$
3) $W \cong_{TOP} X_0(J)$ for some J slice

(if just want to recover Levine-Lidman, can omit)

Suppose W has P.L. spine.

> **Lemma 1:** $X \simeq S^2$ has a spine $\iff \exists$ some K, n s.t. $X_n(K) \underset{sm}{\overset{\phi}{\hookrightarrow}} X$ w/ ϕ giving a homotopy equivalence and $X \smallsetminus \phi(\mathring{X}_n(K))$ a homology cobordism.

Then $\exists$ K w/ $X_0(K) \underset{sm}{\overset{\phi}{\hookrightarrow}} X$, ϕ gives homotopy equivalence.
$X \underset{sm}{\hookrightarrow} S^4 \Rightarrow X_0(K) \underset{sm}{\hookrightarrow} S^4 \Longrightarrow K$ is slice $\Longrightarrow$
$S^2 \underset{sm}{\hookrightarrow} X_0(K)$ giving h.e. $\Longrightarrow S^2 \underset{sm}{\hookrightarrow} X$ giving h.e.
$\Longrightarrow X$ has smooth spine $\lightning$



How to build W s.t.

0) $Q_W = [0]$, $W \simeq S^2$ — homotopy equivalent

1) W has no smooth spine

2) $W \xhookrightarrow[sm]{} S^4$

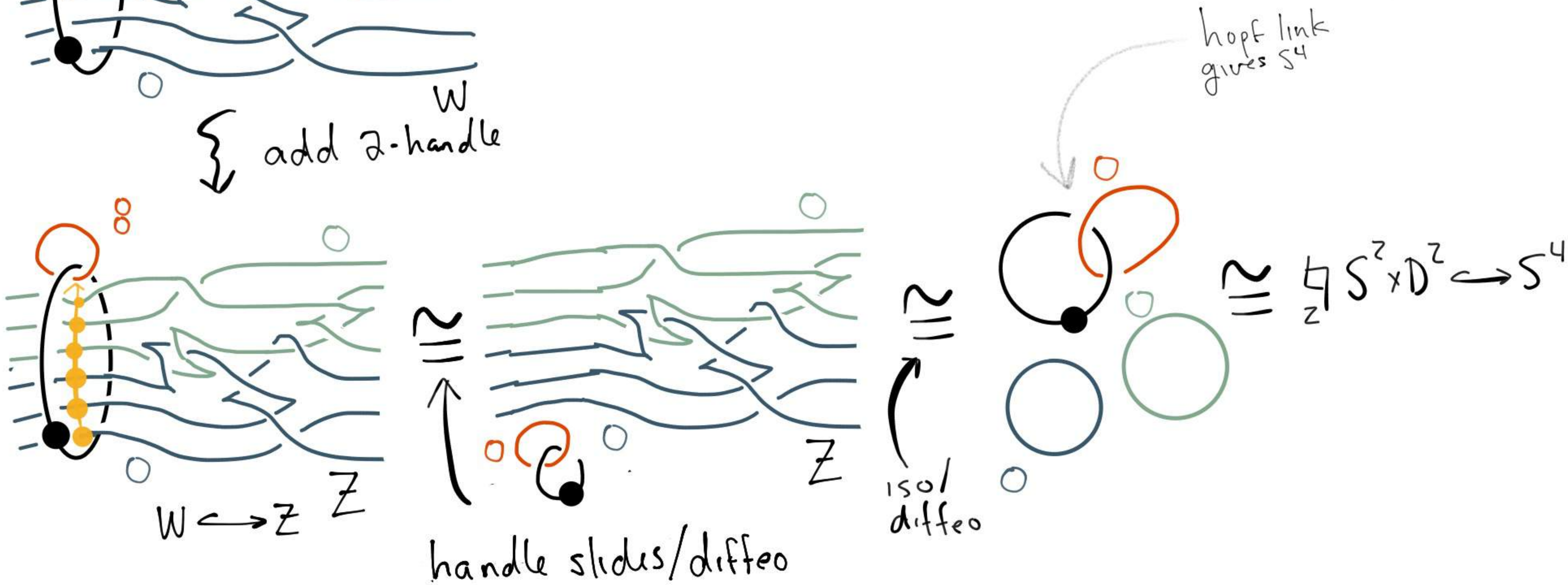
3) $W \cong_{TOP} X_0(J)$ for some J slice

$n = g = 0$:

0) obvious or routine calculation

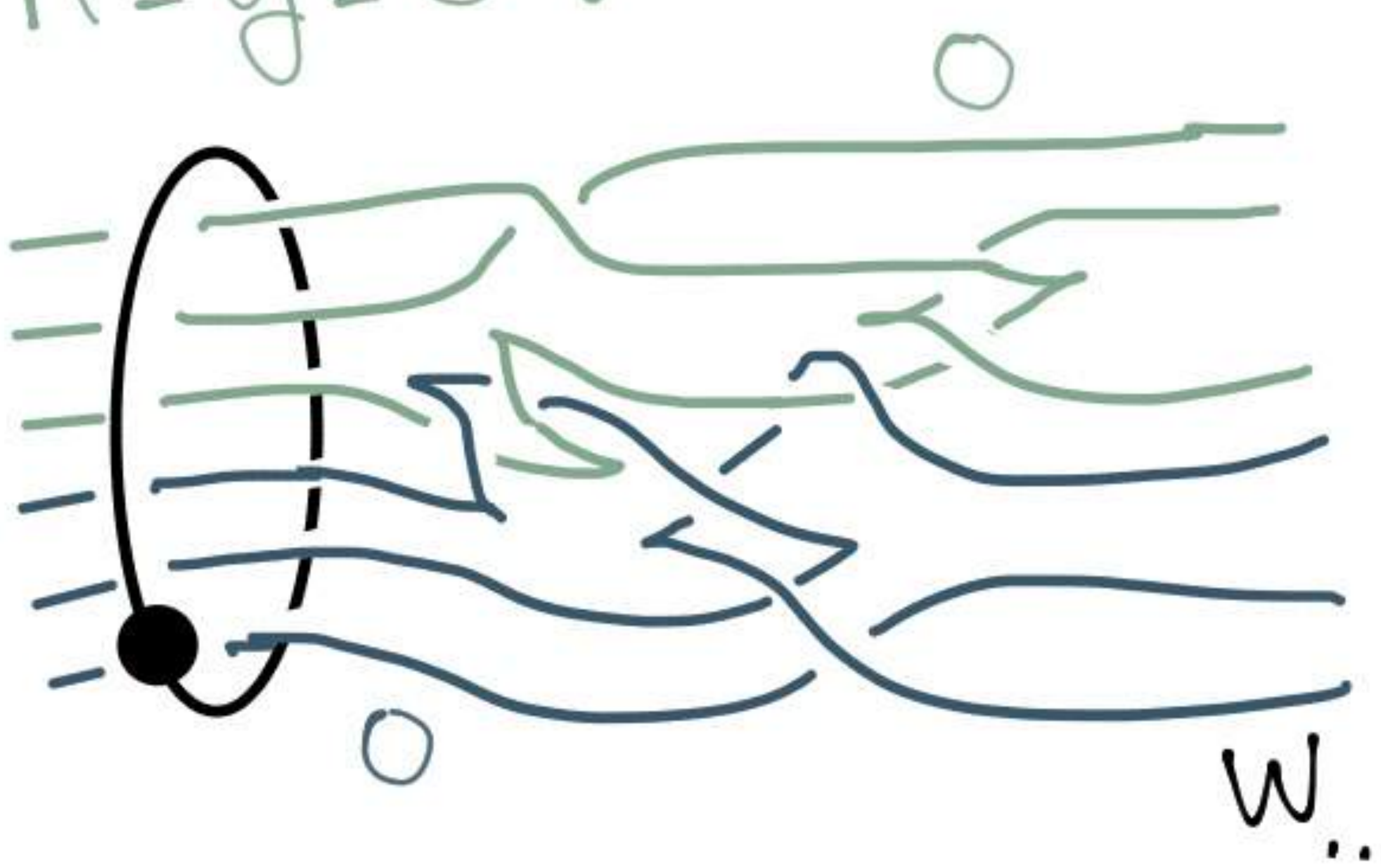
2) build $W \xhookrightarrow[sm]{} S^4$ by hand

add 2-handle



How to build W s.t. $\left\{ \begin{array}{l} \checkmark \; 0)\; Q_W = [0],\; W \simeq S^2 \quad \text{homotopy equivalent} \\ \checkmark \; 1)\; W \text{ has no smooth spine} \\ \checkmark \; 2)\; W \xhookrightarrow[sm]{} S^4 \\ \quad 3)\; W \cong_{TOP} X_0(J) \text{ for some } J \text{ slice} \end{array} \right.$

$n = g = 0$:



1) Defn/Thm (Eliashberg): W^4 admits Stein structure

$\iff$ W admits a handle diagram w/ no 3-handles which satisfies a framing condition on 2-handles

Thm (Lisca-Matic, '98): If $\Sigma^2 \xhookrightarrow[sm]{} X^4$ w/ X Stein then

$$[\Sigma]\cdot[\Sigma] \leq 2g(\Sigma) - 2$$

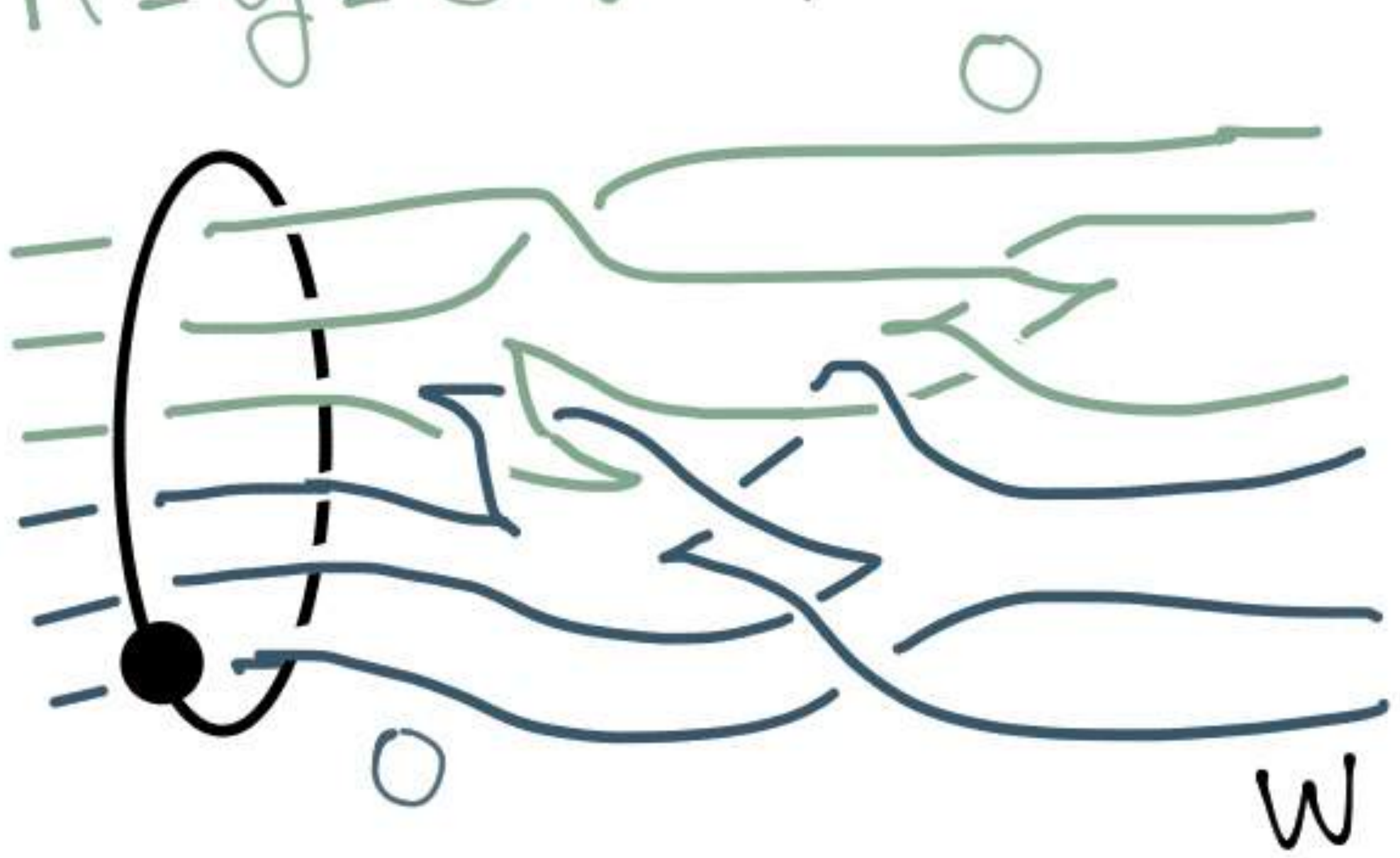
hence, any $\Sigma^2 \hookrightarrow W$ such that $[\Sigma]$ gen H_2 has

$0 \leq g(\Sigma) - 2 \Rightarrow \Sigma$ not S^2

How to build W s.t.

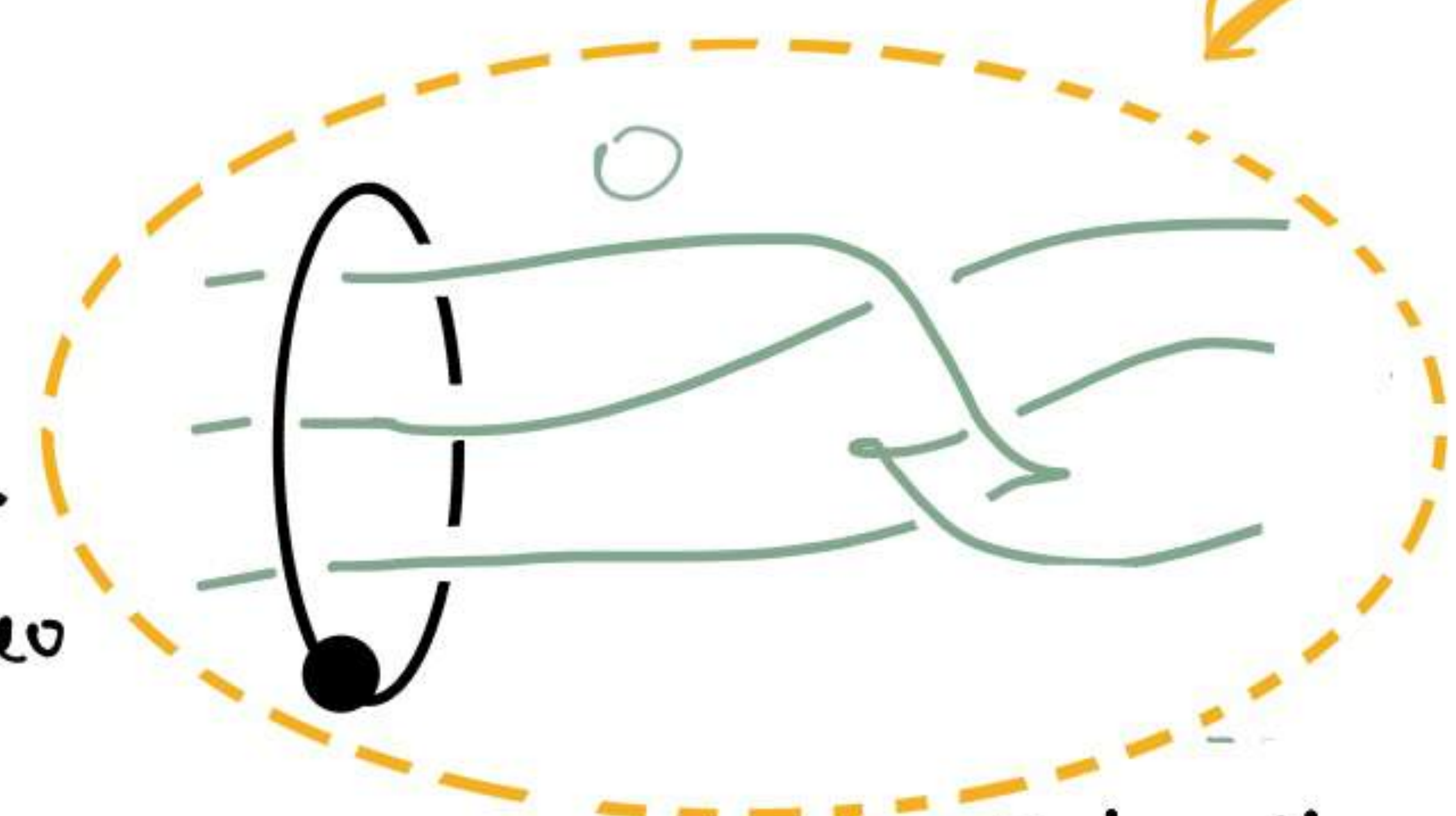
- ✓ 0) $Q_W = [0]$, $W \simeq S^2$ ← homotopy equivalent
- ✓ 1) W has no smooth spine
- ✓ 2) $W \xhookrightarrow[sm]{} S^4$
- 3) $W \cong_{TOP} X_0(J)$ for some J slice

$n = g = 0$: 3)



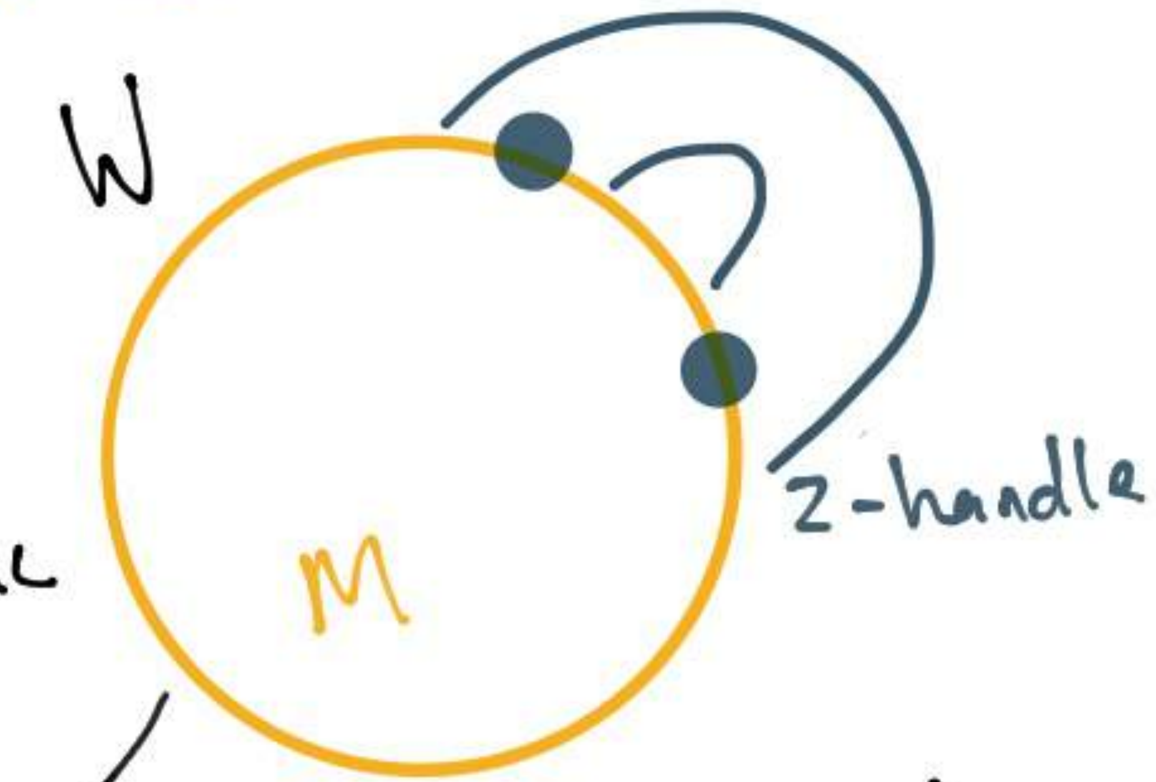
$\cong$ diffeo

contractible M "the Mazur manifold"



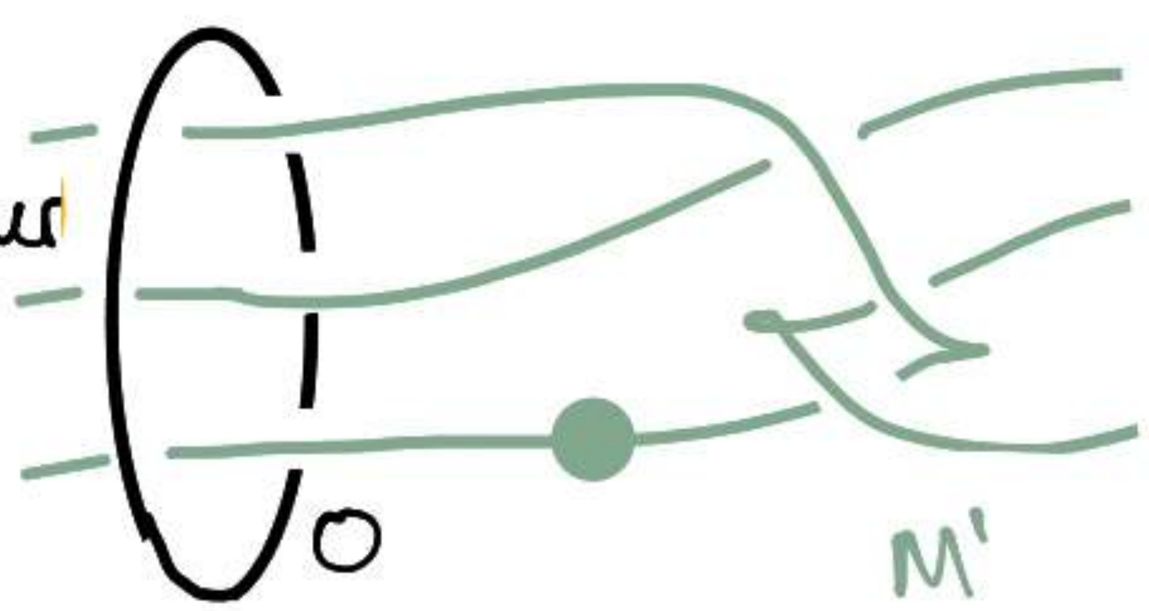
$\cup$ blue 2-handle

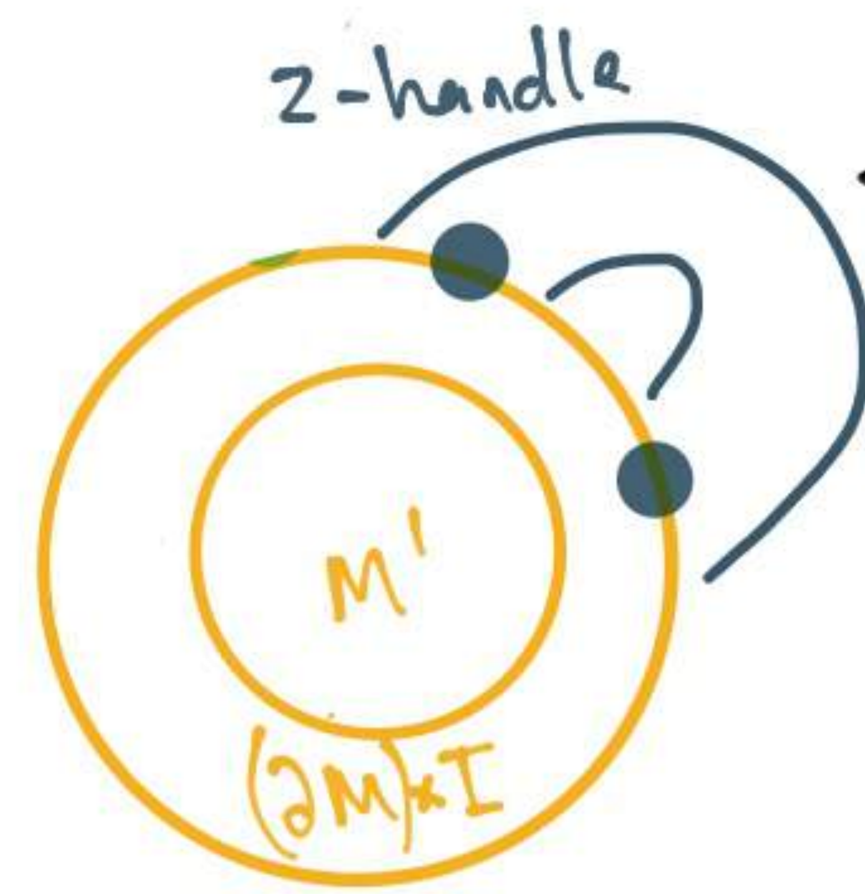
$\cong$ schematic



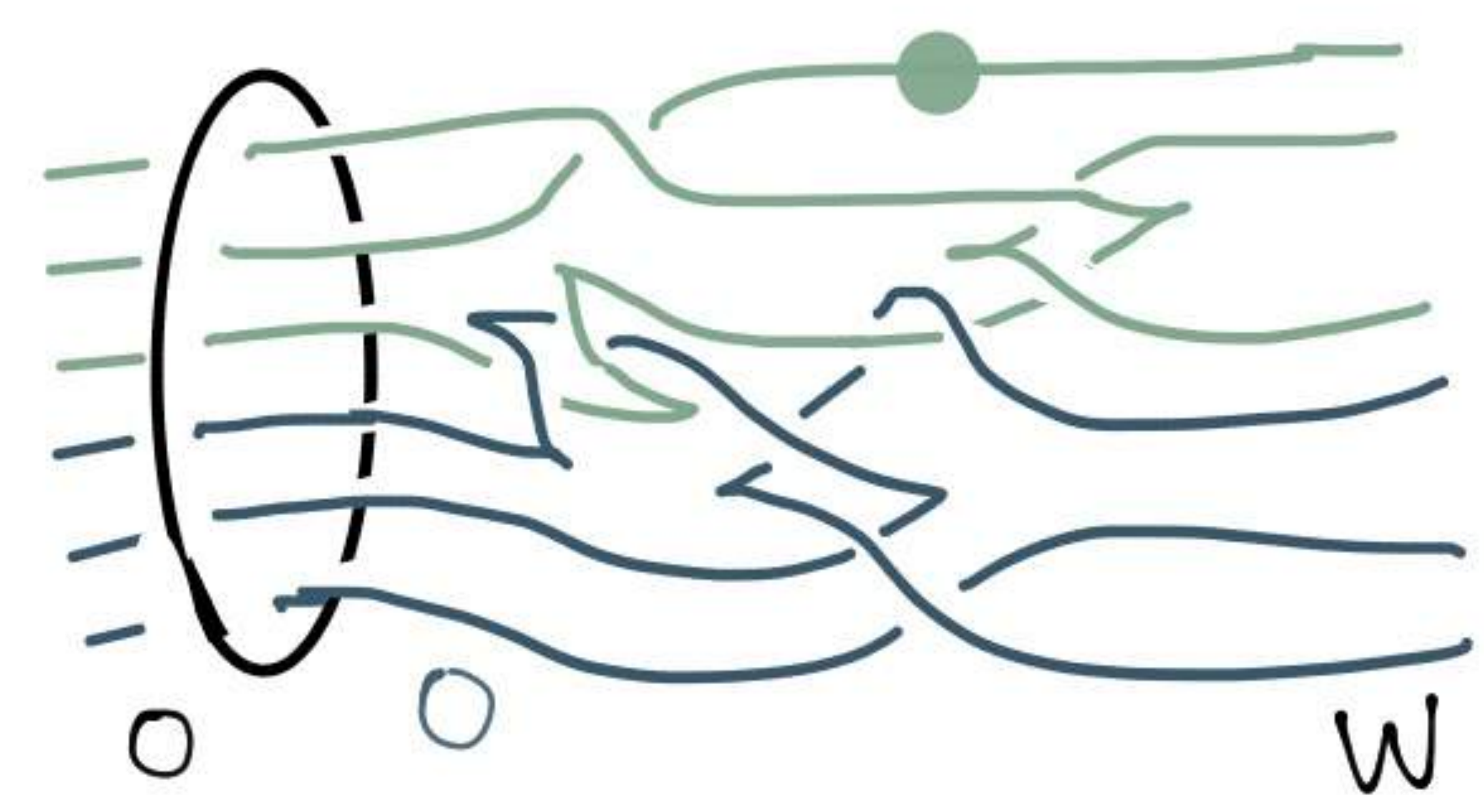
Replace M with M'
Freedman $\Rightarrow M \cong_{TOP} M'$
rel boundary, hence $W \cong_{TOP} W'$

Fact: ∂M admits another contractible filling M'





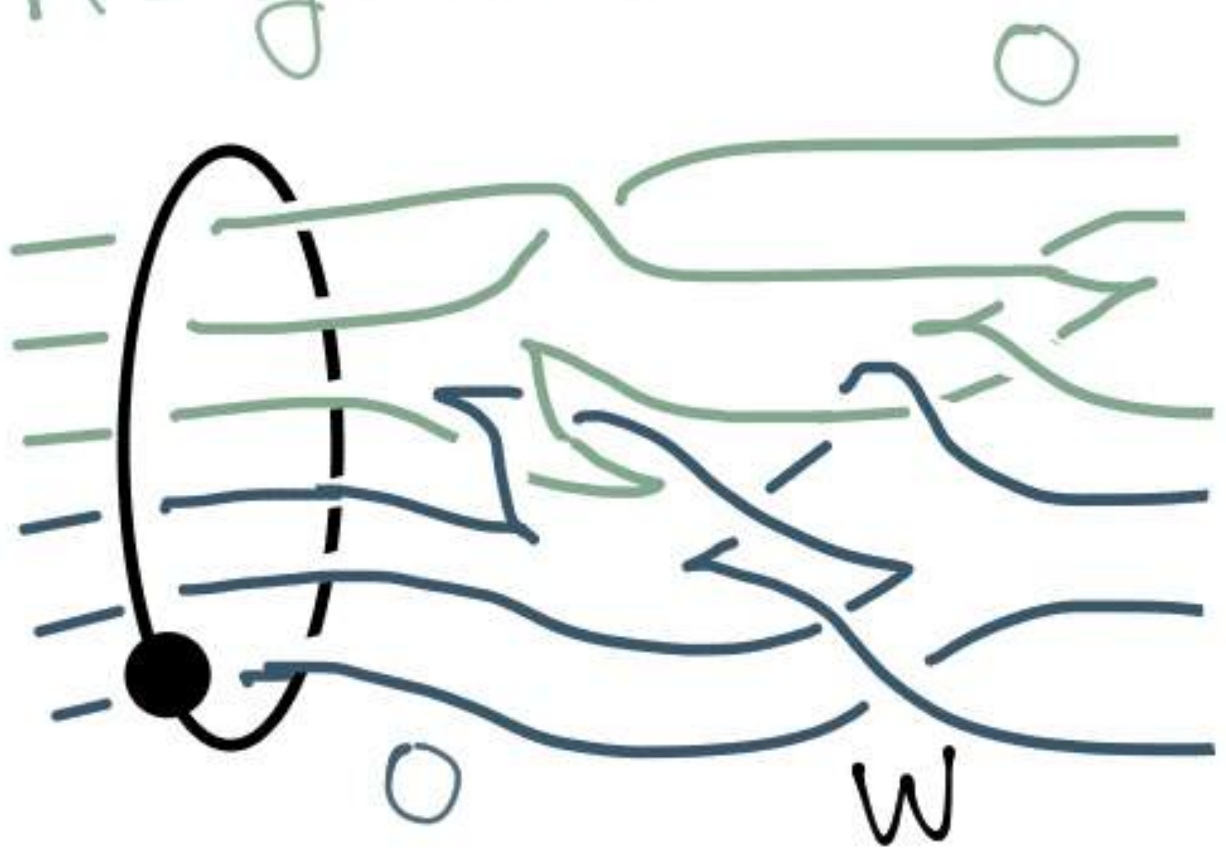
$\cong$



How to build W s.t.

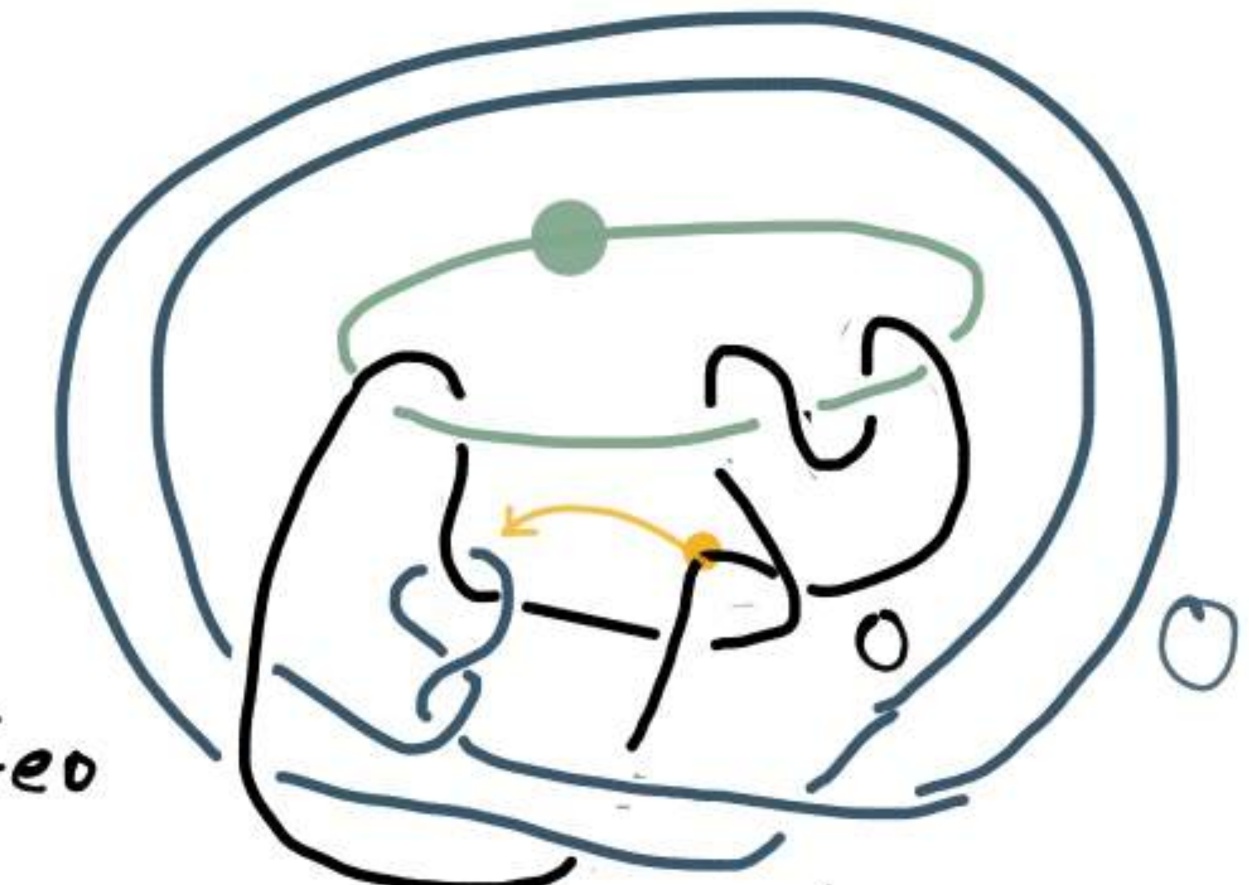
- ✓ 0) $Q_W = [0]$, $W \simeq S^2$ homotopy equivalent
- ✓ 1) W has no smooth spine
- ✓ 2) $W \underset{sm}{\hookrightarrow} S^4$
- 3) $W \cong_{TOP} X_0(J)$ for some J slice

$n=g=0$: 3) $W \cong_{TOP} W'$

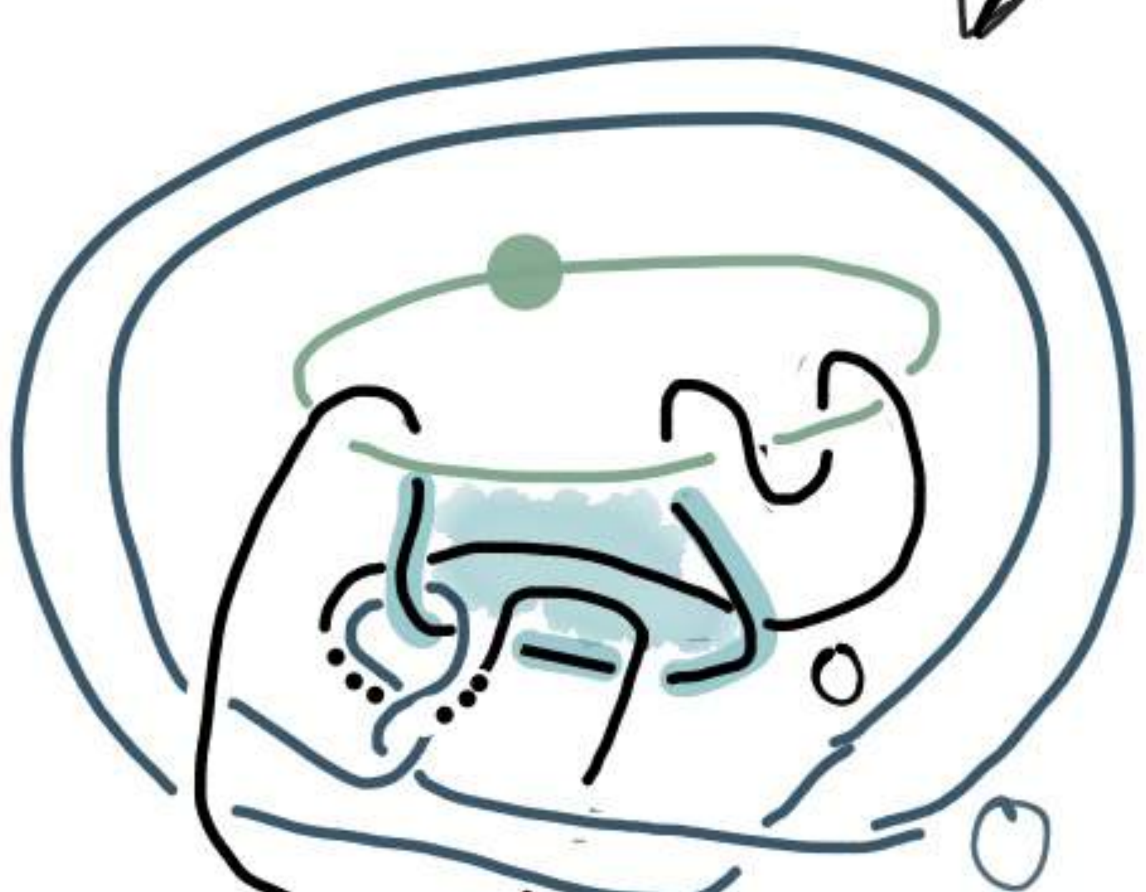


$\cong$ homeo

$\cong$ iso/diffeo



handle slide/diffeo

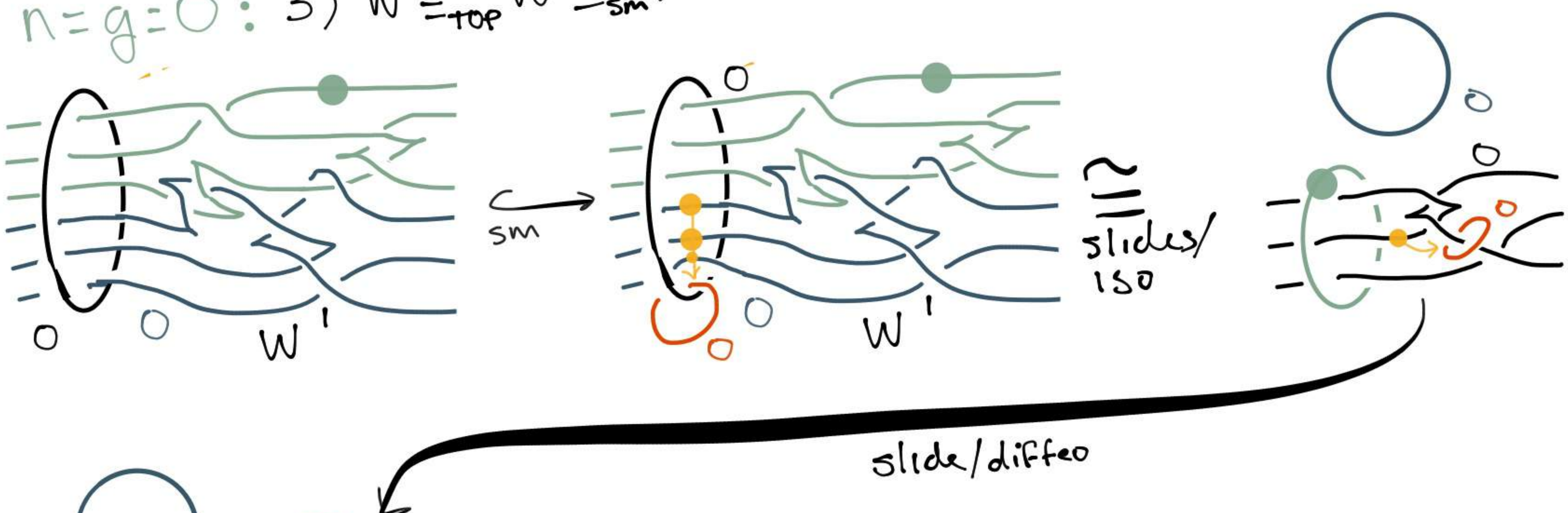


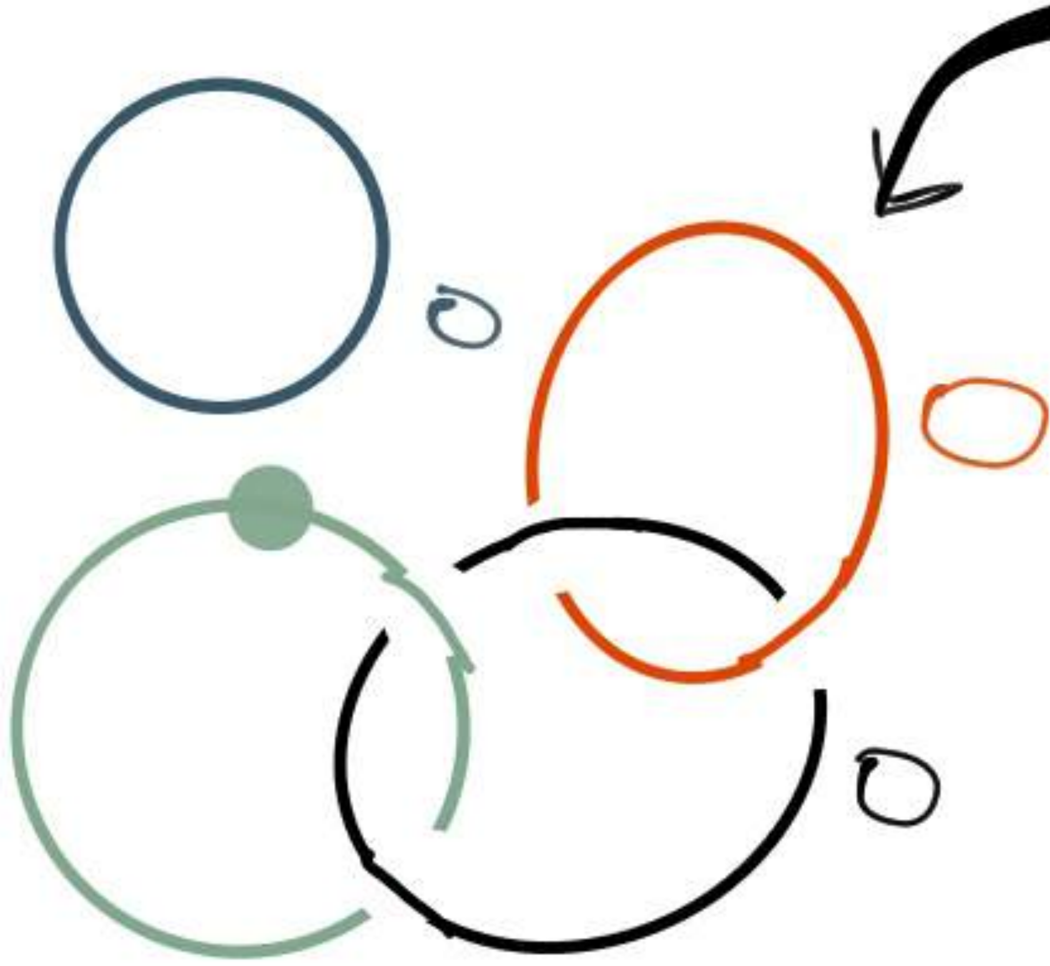
black ∪ green a cancelling 1-2 pair $\Rightarrow W' \cong_{sm} X_0(K_B)$

How to build W s.t.

- ✓ 0) $Q_W = [0]$, $W \simeq S^2$ homotopy equivalent
- ✓ 1) W has no smooth spine
- ✓ 2) $W \underset{sm}{\hookrightarrow} S^4$
- ✓ 3) $W \cong_{TOP} X_0(J)$ for some J slice

$n = g = 0$: 3) $W \cong_{TOP} W' \cong_{sm} X_0(K_B)$





$\underset{\text{diffeo}}{\cong} \bigsqcup_2 S^2 \times D^2 \underset{sm}{\hookrightarrow} S^4$. Trace embedding lemma $\Rightarrow K_B$ slice

How to build W s.t.

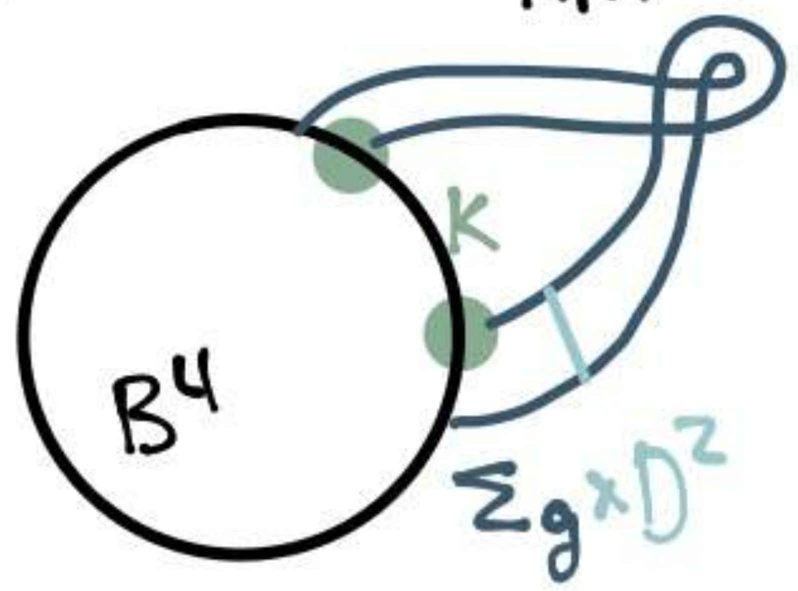
0) $Q_W = [0]$, $W \simeq S^2$ homotopy equivalent

1) W has no smooth spine

2) $W \underset{sm}{\hookrightarrow} S^4$

3) $W \cong_{TOP} X_0(J)$ for some J slice

$g>0$: $X_n^g(K) := B^4 \cup_{K,n} (\mathring{\Sigma}_g \times D^2)$



generalized trace embedding lemma: $g_4(K) \leq g$

$\iff X_0^g(K) \underset{sm}{\hookrightarrow} S^4$

$n>0$: generalized trace embedding lemma: $X_n(K) \underset{sm}{\hookrightarrow} W^4 \iff$

$-K \subseteq \partial(W - \mathring{B}^4)$ bounds $D^2 \underset{sm}{\hookrightarrow} W$ w/ $[D]\cdot[D] = n$