SEMINAR WISE23/24: THE DIRECT SUMMAND CONJECTURE

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The following is the schedule for the HioB seminar in the winter term of 2023/24 at the University of Regensburg. Our goal will be to understand the *direct summand conjecture*, which was conjectured by Hochster in 1969 [Hoc73] and proven by André in 2016 [And16a].

Theorem 0.0.1 (**DSC**)**.** *Let $i: A \to B$ be a finite extension of notherian rings such that A is regular. Then i is split as a map of A-modules.*

Anybody interested in participating please contact one of the organisers at "firstname.lastname@ur.de".

This conjecture, being very easy to state, seems to contain deep mathematical content. For example it plays a central role in the so called *homological conjectures* from homological algebra. For more on this the reader is invited to consult [Hoc00].

To understand a bit better what will happen in the seminar let us outline vaguely how the proof will go in the case[1] where A has the form

$$W(k)\widehat{[[x_1, \dots, x_d]]}$$

with k a perfect field of characteristic p and the completion being p-adic (following [Bha17b]). The ideas of proving the direct summand conjecture circle around Faltings' *almost purity* theorem [Fal02], of which we state a version now

Theorem 0.0.2 (**almost purity**)**.** *Consider a perfectoid affinoid K-algebra (R, R^+), where K is perfectoid field. Then for any finite étale R-algebra S the integral closure of R^+ in S is almost finite étale over R^+. Furthermore this construction gives an equivalence*

$$\{\textit{finite étale over } R\} \simeq \{\textit{almost finite étale over } R^+\}$$

with inverse given by inverting the pseudo-uniformizer.

To see why this helps in our situation, let us first assume that $A[\frac{1}{p}] \to B[\frac{1}{p}]$ is étale. In this case, if we write A_∞ for the p-adic completion of

$$A[x_1^{\frac{1}{p^\infty}}, p^{\frac{1}{p^\infty}}]$$

the almost purity theorem tells us that the integral closure B_∞ of B in $B \otimes_A A_\infty[\frac{1}{p}]$ is almost finite étale over A_∞. Concretely the almost here means that the obstruction of the map $A_\infty \to B_\infty$ being finite étale, such as the Ext^1-class measuring the failure of this map to split, is killed by $p^{\frac{1}{p^k}}$ for all $k \geqslant 0$. Using that the map $A \to A_\infty$ is faithfully flat one then is able to descent this almost splitting to and actual splitting of $A \to B$.

To make this idea work in general André came up with the following construction [And16a]:

[1] Actually there exists a theorem of Hochster showing that this is the only case left to show [Hoc73]

Theorem 0.0.3 (**André flatness lemma**). *For any $g \in A$, there exists a map $A_\infty \to \tilde{A}_\infty$ of integral perfectoid algebras which is almost faithfully flat modulo p, such that $g \in A$ admits a compatible system of p-power roots $g^{\frac{1}{p^k}}$ in $\tilde{A}_\infty$.*

The proof of this theorem heavily relies on perfectoid geometry and in order to apply this for the direct summand conjecture one chooses g, such that $A[\frac{1}{g}] \to B[\frac{1}{g}]$ is étale. The faithfully flatness assertion in the theorem tells us that we have reduced the proof to constructing an almost splitting of the map

$$\tilde{A}_\infty \to B \otimes_A \tilde{A}_\infty$$

Here we will not follow the original argument of André, which uses the perfectoid Abhyankar lemma [And16b], but a method of Bhatt [Bha17b], which also can be used to prove a derived version of the direct summand conjecture:

Theorem 0.0.4 (**derived DSC**). *Let A be a regular notherian ring and $X \to \operatorname{Spec}(A)$ a surjective proper map of schemes. Then the map*

$$A \to R\Gamma(X, \mathcal{O}_X)$$

splits in $\mathcal{D}(A)$.

To construct this almost splitting we use perfectoid geometry again. More concretely we consider the ind-system of rational subsets

$$U_n := \{x \in X \,|\, |p^n| \leqslant |g(x)|\}$$

on the perfectoid space X associated to $\tilde{A}_\infty$. This gives a pro-system of bounded functions

$$\mathcal{O}_X^+(U_n)$$

such that g divides p^n in $\mathcal{O}_X^+(U_n)$ and thus we have that the base change of the map $A \to B$ to $\mathcal{O}_X^+(U_n)[\frac{1}{p}]$ is finite étale for all n. Now we can use the almost purity theorem to construct an almost splitting of the base change to $\mathcal{O}_X^+(U_n)$ for all n.

To descent this almost splitting to $\tilde{A}_\infty$, Bhatt proves a quantitative version of Scholze's Riemann extension theorem, which it self is a perfectoid version of the classical Riemann extension theorem.

The reason we already outlined the prove here in the schedule is that we wanted to point out that every step in the proof of the direct summand conjecture heavily relies on the theory of perfectoid geometry introduced by Scholze [Sch11]. In particular the main aim of this seminar will be to learn the basics of this theory, in order to understand the statements and poofs of the above theorems.

Our main reference will be lecture notes of a course given by Bhargav Bhatt on this topic [Bha17a]. But as always it is definitely useful to also consult other references such as the original text on perfectoid spaces by Scholze [Sch11], the book on adic spaces by Huber [Hub13] and lecture notes by Wedhorn [Wed], Morel [Mor] or Weinstein [Wei].

1. Our base fields

1.1. **Tilting of perfectoid fields (16.10).** Define *perfections* [Bha17a][2.0.1] and discuss there basic properties. If possible prove [Bha17a][2.0.10] (otherwise just record it for later use). Define *perfectoid fields* [Bha17a][3.1.1] and give some examples of such. The hart of the talk should concert tilting for perfectoid fields, that is prove [Bha17a][3.2.4+3.2.6+3.2.10].

2. Almost mathematics

2.1. **Almost commutative algebra (23.10).** Construct almost modules and explain the basic situation [Bha17a][4.1]. In particular explain [Bha17a][4.1.7]. Maybe a comparison to the topological situation would be helpful [Bha17a][4.1.8]. Define the standard notions used, such as *almost zero*, *almost flat* and *almost projective* [Bha17a][4.2]. Here an explanation, why the naive version of almost projectivity is not the correct one would be useful [Bha17a][4.2.5]. Explain how to derive functors in this setting [Bha17a][4.2.3] and discuss [Bha17a][4.4.1].

2.2. **Almost purity in characteristic p (30.10).** Explain almostification of algebras and its adjoints [Bha17a][4.2.8]. Define almost finite étale extensions [Bha17a][4.3.1] and give some example. Explain how a finite étale ring map gives rise to an explicit direct summand of a free module [Bha17a][4.3.3]. The main aim of this talk will be to explain the proof of *almost purity* in positive characteristic [Bha17a][4.3.4+4.3.6], on the way you should also explain how one obtains an almost splitting in the situation of the almost purity theorem [Bha17a][4.3.5]. End the talk with explaining [Bha17a][4.3.8].

3. Perfectoid Spaces

3.1. **An algebraic description of Banach algebras (06.11).** Define uniform Banach algebras over a non-archimedean field [Bha17a][5.2.1+5.2.4] and explain the several algebraic descriptions of the obtained category given in [Bha17a][5]. That is prove [Bha17a][5.2.5+5.2.6].

3.2. **The cotangent complex (13.11).** Define the cotangent complex and discuss its basic properties [Bha17a][page 41]. Discuss a bit of deformation theory and explain how it implies the *Infinitesimal invariance of formally étale rings* [Bha17a][6.1.3]. This is explained in [Elm][Lecture 3] but you can also for example consult [Ill71]. If time permits you can prove an almost variant of the above theorem [GR02][5.3.27]. To finish you should explain why perfect rings have trivial cotangent complex [Bha17a][6.1.4].

3.3. **Perfectoid algebras (20.11).** Define all versions of perfectoid algebras given in [Bha17a][6.2.1]. Explain [Bha17a][6.2.3] for later use. In this talk we will formulate the *almost purity theorem* and use the tilting equivalence to deduce it for characteristic 0 perfectoid fields from the characteristic p case. First discuss tilting of perfectoid algebras [Bha17a][6.2.5+6.2.6+6.2.7]. Then state the *almost purity theorem* and prove it in the case of fields [Bha17a][6.2.10]. On the way you should explain how to compute colimits and limits in perfectoid algebras [Bha17a][6.2.9] and record [Bha17a][6.2.13] for later use.

3.4. **Adic spaces I: Affinoid Tate rings (27.11).** Define *Tate rings* [Bha17a][7.1.1] and the relevant notions concerning those [Bha17a][Page 54/55]. Explain [Bha17a][7.1.4]. Then define *affinoid Tate rings* [Bha17a][7.2.1] and explain when they are called *complete, henselian* and *Zariski* [Bha17a][7.2.2]. Give some characterisations for these notions [Bha17a][7.2.3] and explain how these compare to one and another [Bha17a][7.2.5]. To finish show that the inclusion of uniform affinoid Tate rings into all affinoid Tate rings admits a left adjoint and that completions of uniform affinoid Tate rings are again uniform [Bha17a][7.2.6 (2.)+(7.)].

3.5. **Adic spaces II: Affinoid adic spaces (04.12).** Define the adic spectrum of an affinoid Tate ring [Bha17a][7.3.1]. Define the *kernel map* and *microbial valuation rings* [Bha17a][7.3.3+7.3.4]. Explain how to construct such valuation rings [Bha17a][7.3.6] and how one can describe the adic spectrum of an affinoid using those [Bha17a][7.3.7]. Show some basic properties of the adic spectrum of an affinoid [Bha17a][7.3.10]. The rest of your time you can use to explain as much as you can of the proof that the adic spectrum of an affinoid is spectral [Bha17a][7.4.1]. In particular you should discuss [Bha17a][7.4.3+7.4.4+7.4.5].

3.6. **Adic spaces III: The structure presheaf (11.12).** Explain how one should understand functions on an affinoid adic space [Bha17a][7.5.1+7.5.3]. Define stalks of an affinoid adic space and describe there local rings [Bha17a][7.5.4+7.5.5]. Explain how to understand Zariski closed subsets of an affinoid adic space [Bha17a][7.5.8]. Define the structure presheaf on an affinoid adic space and what it means for an affinoid Tate ring to be sheafy [Bha17a][7.5.11+7.5.12]. To end the talk you can finally define adic spaces [Bha17a][7.5.14].

3.7. **Tilting rational subsets (18.12).** Define *perfectoid affinoid* algebras [Bha17a][9.1.1] and prove the tilting correspondence for those [Bha17a][9.1.2]. Explain what are perfectoid affinoid fields [Bha17a][9.1.3]. The main goal of this talk is to understand how one extends the tilting correspondence to rational subsets. That is prove [Bha17a][9.2.2] and [Bha17a][9.2.7], in particular give a more detailed description of Huber's presheaf in this situation [Bha17a][9.2.3+9.2.5].

3.8. **Tate acyclicity for perfectoids (08.01).** The goal of this talk will be to understand why perfectoid affinoids are acyclic with respect to the structure presheaf [Bha17a][9.3.1]. Define the notion of *algebraically admissible* [Bha17a][9.3.2] and prove *Tate acyclicity* for classical affinoid algebras [Bha17a][9.3.3]. Then deduce *Tate acyclicity* for p-finite perfectoid algebras [Bha17a][9.3.7] and explain how to approximate arbitrary perfectoid algebras [Bha17a][9.3.7]. On the way you should also explain [Bha17a][9.3.6].

4. Proving the direct summand conjecture

4.1. **André's flatness lemma (15.01).** Define *perfectoid spaces* [Bha17a][9.3.9] and state the *Tilting correspondence* in full generality [Bha17a][9.3.10] (at this point we already will have proven it). Prove that perfectoid spaces have fibre products [Bha17a][9.3.13]. Explain how to understand Zariski closed subsets of perfectoids [Bha17a][9.4.1] and use this to prove *André's flatness lemma* [Bha17a][9.4.3].

4.2. **The almost purity theorem (22.01).** In this talk we finish proving *the almost purity theorem*. That is explain the prove of [Bha17a][10.0.1]. To do so start by defining finite étale maps of adic spaces and strongly finite étale maps of perfectoids [Bha17a][10.0.2+10.0.3]. Deduce from [Bha17a][10.0.4] that taking finite étale sites of complete uniform affinoid Tate algebras commutes with filtered colimits [Bha17a][10.0.5]. Explain why a strongly finite étale perfectoid space over an affinoid is again affinoid [Bha17a][10.0.6] and deduce [Bha17a][10.0.7]. Prove (one half of) quasi-coherence of locally free $\mathcal{O}_X$-modules of an affinoid perfectoid [Bha17a][10.0.8], then you are ready to explain finish the proof of almost purity. To finish you should explain how finite étale maps of affinoid perfectoids give rise to almost splittings [Bha17a][10.0.9].

4.3. **The direct summand conjecture (29.01).** Finally we can prove the *direct summand conjecture*. Start by explaining the case in pure characteristic 0 [Bha17a][11.0.3] and then as a motivation you can outline how a prove in pure characteristic p could go [Bha17a][11.1.2]. For the mixed characteristic case define the notion of an almost-pro-isomorphism [Bha17a][11.3.1] and explain [Bha17a][11.3.5]. Prove Bhatt's quantitative version of the Riemann Hebbarkeitssatz in perfectoid geometry [Bha17a][11.2.1] and deduce [Bha17a][11.2.2]. Now you can prove the direct summand conjecture [Bha17b][5.4] and it's derived variant [Bha17b][6.1].

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