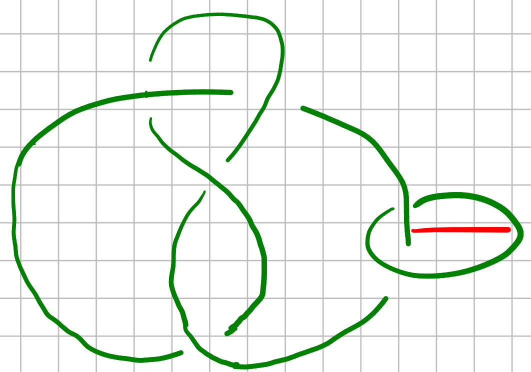
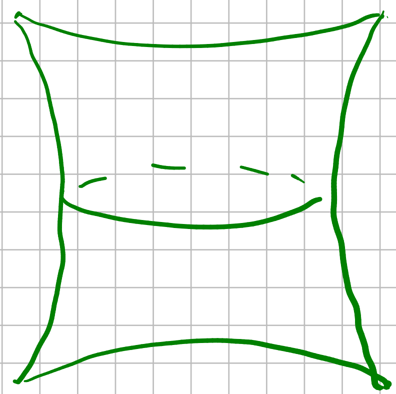
The earring correspondence of the pillowcase





Guillem Cazassus, Oxford

joint with Chris Herald, Paul Kirk, and Artem Kotelskiy.

• Instanton homology

→ **Original version** (Floer)

Y : 3-manifold (rational homology sphere)

↪ $I_*(Y) \approx$ "Morse homology of the Chern-Simons functional"

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• $\partial : CI_*(Y) \to CI_*(Y)$ counts "Anti self-dual" instantons on $Y \times \mathbb{R}$: $\mathbb{A}$ / $F_{\mathbb{A}} + * F_{\mathbb{A}} = 0$

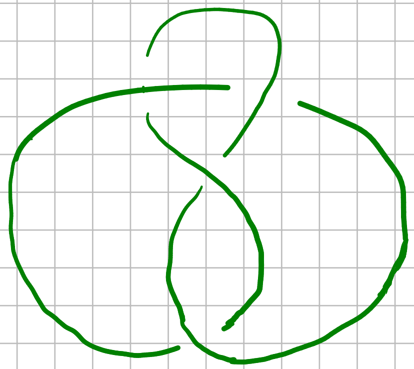
$\mathbb{A}$: connexion on $Y \times \mathbb{R}$ $\updownarrow$ $\{A_t\}_t$: family of connexions on Y

$\Longleftrightarrow$ gradient trajectories of CS

$$\partial_t A_t = - * F_{A_t} = - \nabla CS_{A_t}$$

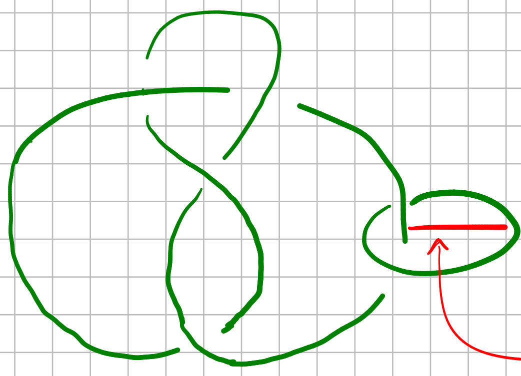
→ Singular Instanton Homology (Kronheimer-Mrowka)

$K \subset S^3$ knot



$\rightsquigarrow$ adding an "earring"

$K^\natural$



"w_2 arc"

$\rightsquigarrow I^\natural(K)$

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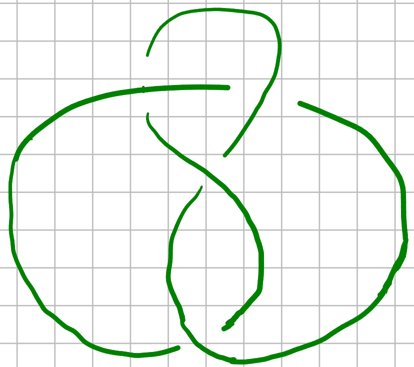

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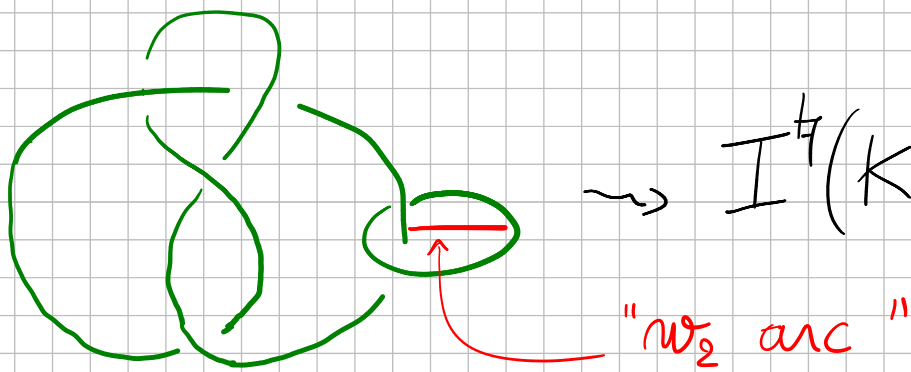


$\rightsquigarrow I^{\natural}(K)$

Similar construction applied to connexions on $S^3 \setminus K^{\natural}$ satisfying:

- $\circlearrowleft \; Hol_A \in C \subset SU(2)$ "traceless condition"
- $\circlearrowleft \; Hol_A = -1 \in SU(2)$ "w_2 condition"
  ($= w_2(P)$, P : $SO(3)$-bundle)

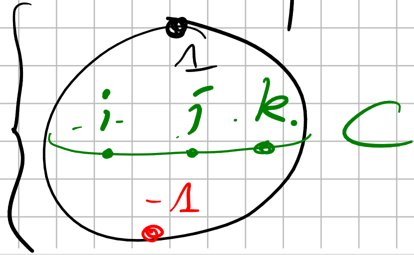
$SU(2) = \{ q \in \mathbb{H}, |q| = 1 \}$

$\mathfrak{su}(2) = \{ q \in \mathbb{H}, \mathrm{Re}\, q = 0 \}$

$C = SU(2) \cap \mathfrak{su}(2) \simeq S^2$

"traceless sphere"

$SU(2)$



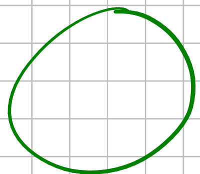
Remarks:

→ Other versions: links, annular links ($\subset S^1 \times D^2$) tangles, graphs, webs, foams ...

→ Spectral seq. $\widetilde{Kh}(mK) \Rightarrow I^{4}(K)$ (Kronheimer-Mrowka)

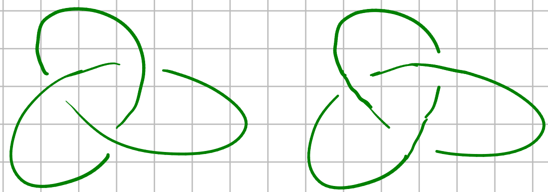
↳ $\widetilde{Kh}$ detects:

× The unknot



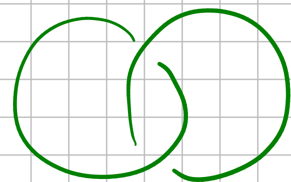
(Kronheimer-Mrowka)

× The trefoil knots



(Baldwin-Sivek)

× the Hopf link



(Baldwin-Sivek-Xie)

→ (Conjecturally) Proof of the four-color theorem (Kronheimer-Mrowka)

→ Pillowcase homology (Hedden–Herald–Kirk)

Goal: compute $I^\natural(K)$.

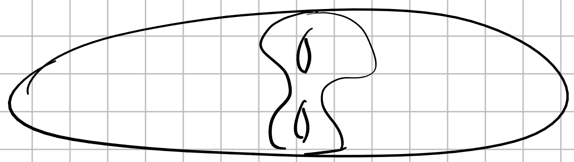
• **Atiyah–Floer conjecture:**

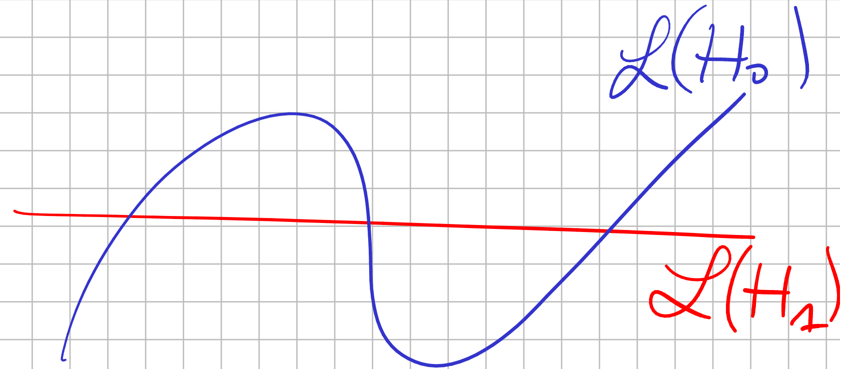
3D object ↓ (Y) — 3D object with boundary (H_0, H_1) — 2D object (Σ)

$$Y = H_0 \cup_\Sigma H_1 \quad \rightsquigarrow \quad \mathcal{L}(H_0) \hookrightarrow \mathcal{M}(\Sigma) \hookleftarrow \mathcal{L}(H_1)$$

moduli spaces

symplectic manifold ($\mathcal{M}(\Sigma)$) — Lagrangians ($\mathcal{L}(H_0)$, $\mathcal{L}(H_1)$)





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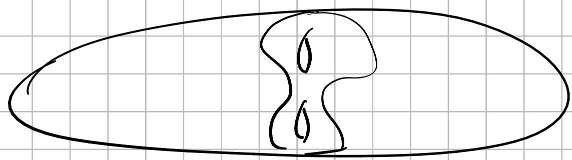
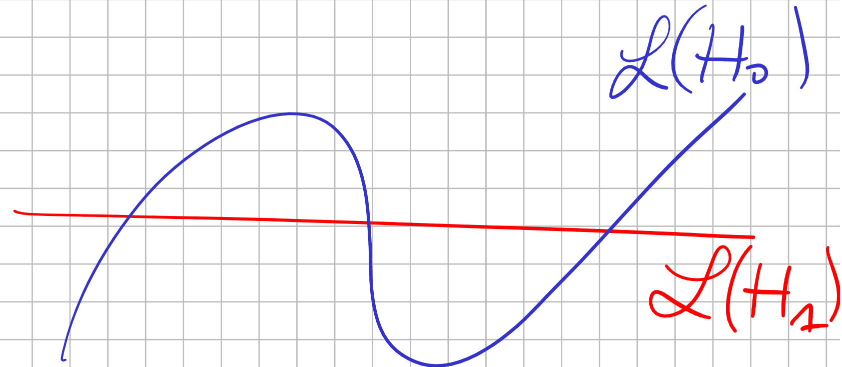
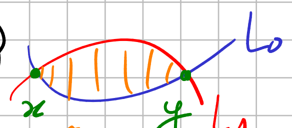




Conj: $I_*(Y) \simeq HF(\mathcal{L}(H_0), \mathcal{L}(H_1))$

Lagrangian Floer Homology

• $CF(L_0, L_1) = \bigoplus_{x \in L_0 \pitchfork L_1} \mathbb{Z}_2 \cdot x$

• $\partial x = \sum_y{}' \# \{$  $\} \cdot y$

pseudo-holom. disc

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3D object → Y $=$ $H_0 \cup_\Sigma H_1$ ← 3D object with boundary; 2D object (Σ)

$\rightsquigarrow$ moduli spaces

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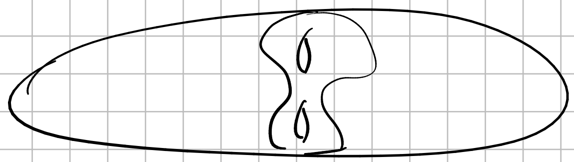

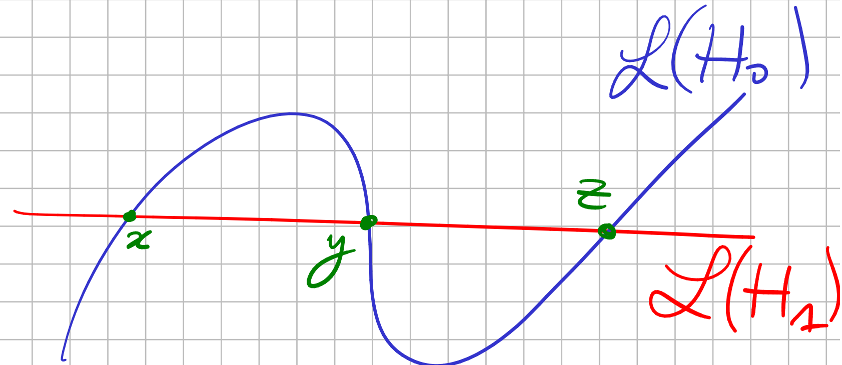

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Lagrangian Floer Homology (→ HF)

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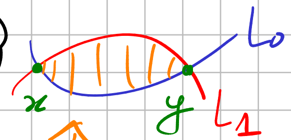
- $\partial x = \sum_y' \# \Big\{$  $\Big\} \cdot y$

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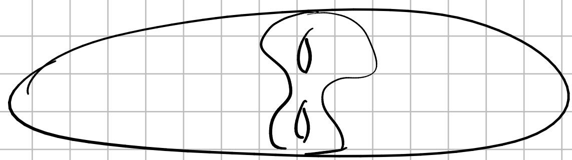

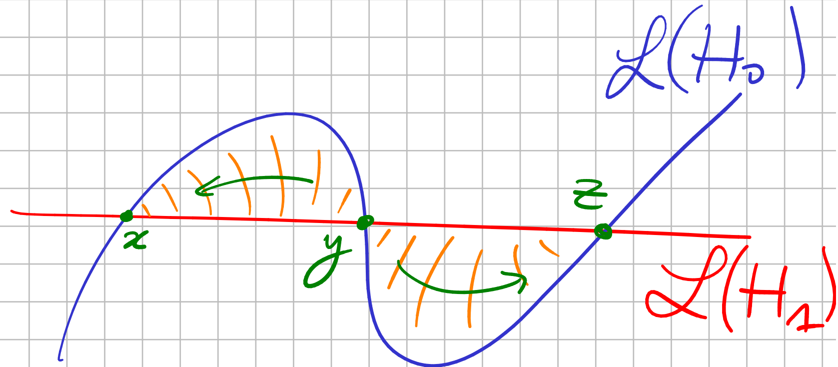

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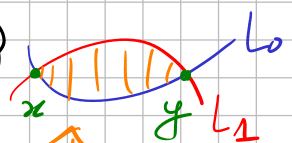
- $\partial x = \sum_y' \# \{ \text{pseudo-holom. disc} \} \cdot y$



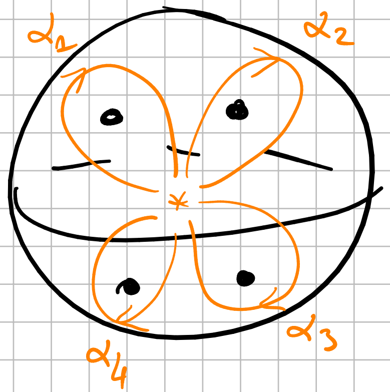
pseudo-holom. disc

$\partial y = x + z$.
$\partial x = \partial z = 0$.

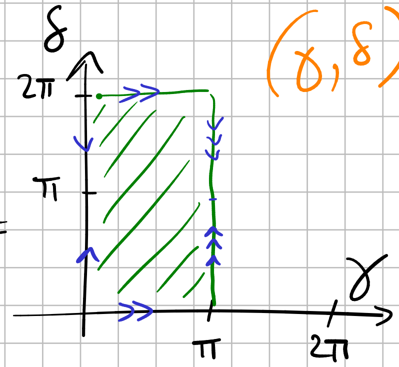
Atiyah-Floer conjecture applied to $I^\#(K)$:
(Hedden-Herald-Kirk)

$$\Sigma_1' = (S^2, 4\,\text{pts}) \rightsquigarrow \mathcal{M}(\Sigma_1') = \{(a_1, a_2, a_3, a_4) \in \mathbb{C}^4 : a_1 a_2 a_3 a_4 = 1\} \big/ SU(2)^{Ad}$$

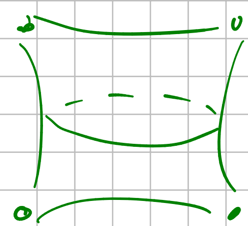
$$(i, e^{\gamma j} i, e^{\delta j} i, e^{(\gamma-\delta)j} i) \;\Uparrow\; (\gamma, \delta)$$



$=$



$\simeq$



$= P.$

"Pillowcase"

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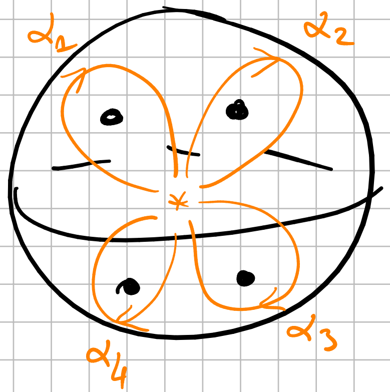

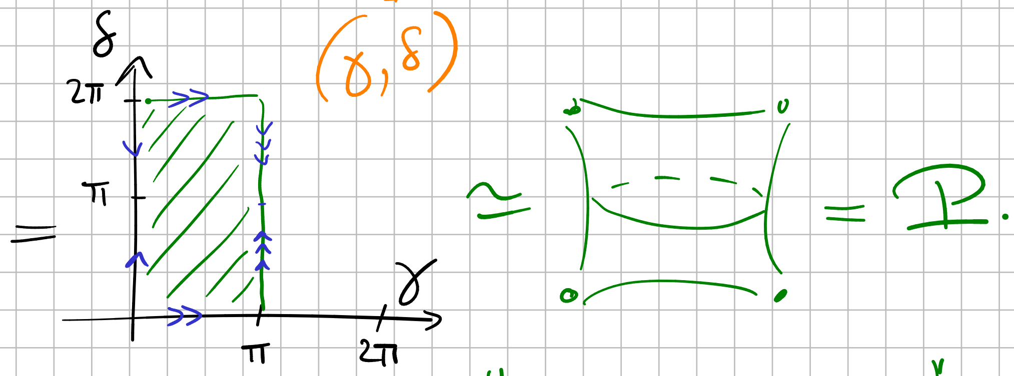

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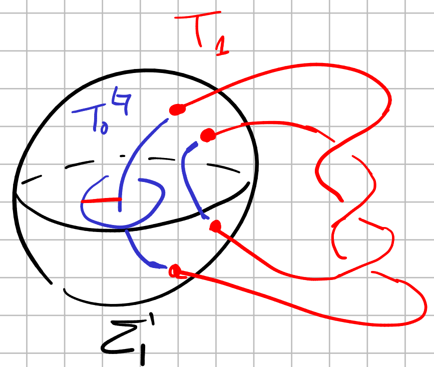
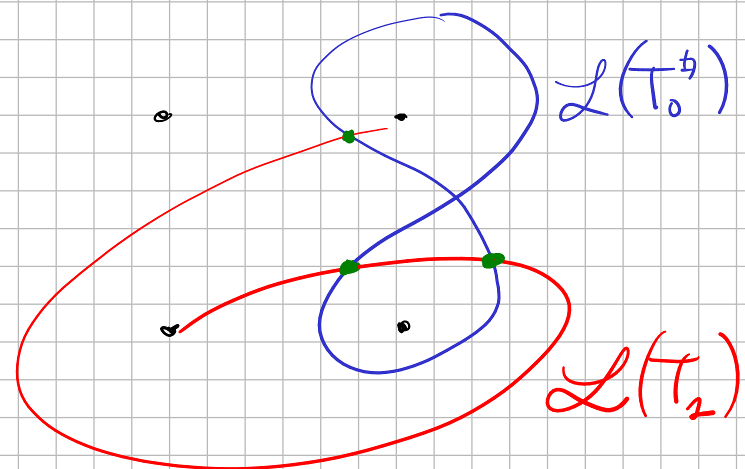
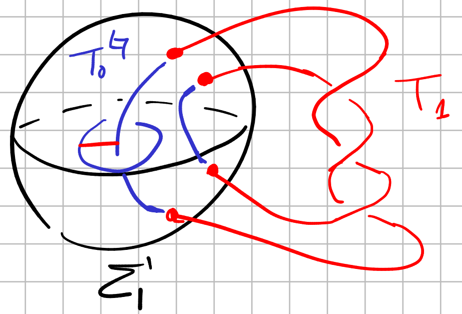


(+ holonomy perturbations) $\rightsquigarrow$





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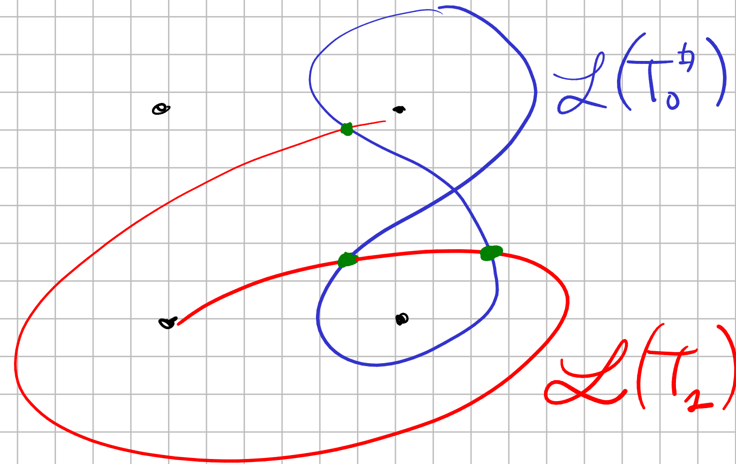
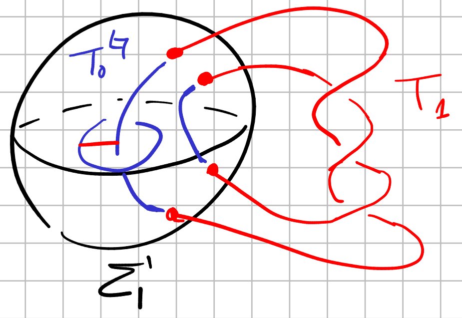
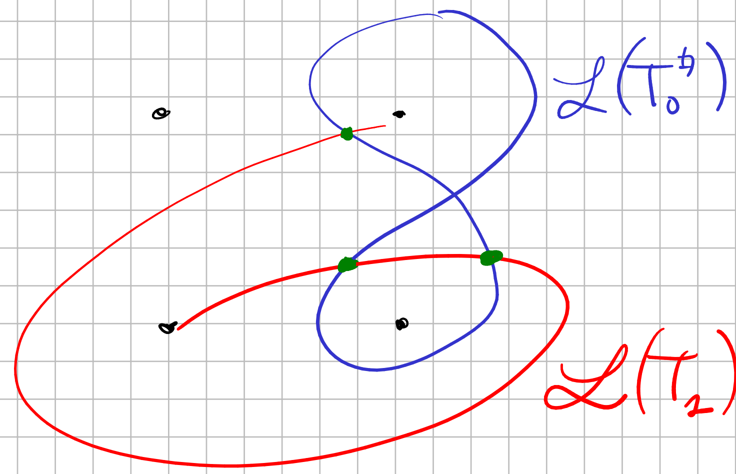


<u>Th</u>: (Hedden – Herald – Kirk)

- For suitable choices of "holonomy perturbations" on $T_0^\natural$ and T_1, $\mathcal{L}(T_0^\natural)$ and $\mathcal{L}(T_1)$ are smooth Lagrangian immersions. Furthermore, $\mathcal{L}(T_0^\natural)$ avoids the singular locus of P.
- The Lagrangian Floer homology group $HF(\mathcal{L}(T_0^\natural), \mathcal{L}(T_1))$ is well-defined (i.e. $\partial^2 = 0$).
- $CI^\natural(K) \simeq CF(\mathcal{L}(T_0^\natural), \mathcal{L}(T_1))$ as vector spaces.



$\widetilde{\longrightarrow}$ (+ holonomy perturbations)



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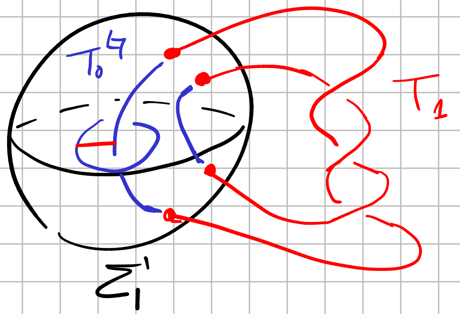

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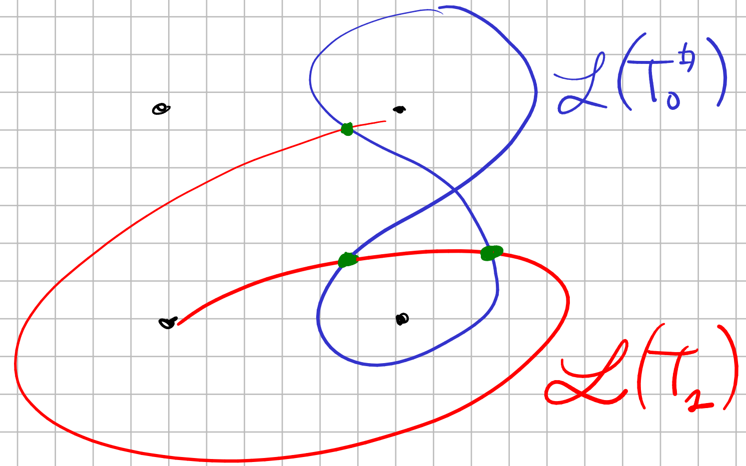

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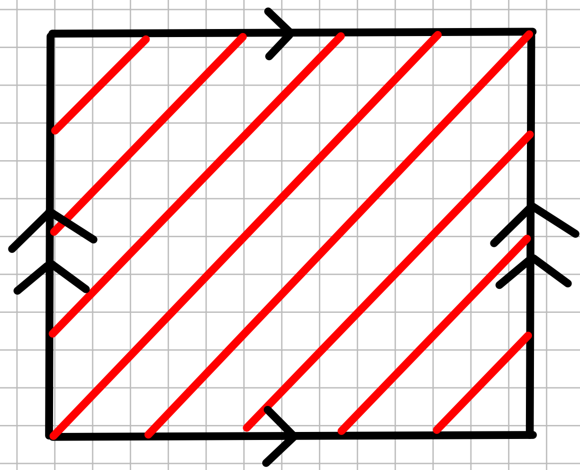
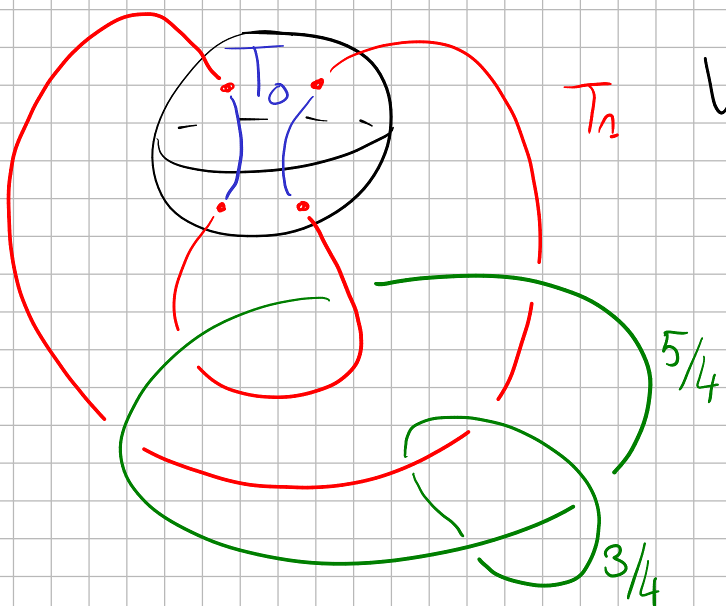
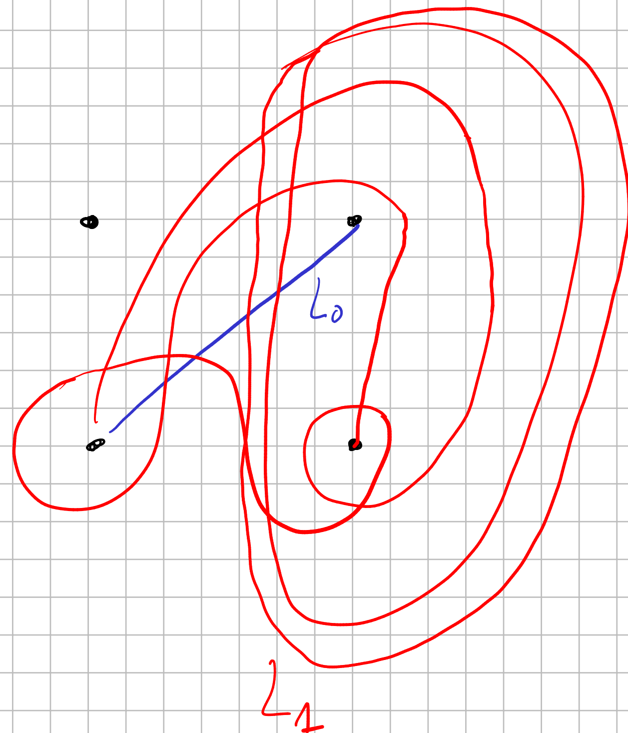
<u>Q</u>: $HF(\mathcal{L}(T_0^\natural), \mathcal{L}(T_1))$ invariant of K? <u>A</u>: No.

Example: The (4,5)-torus knot





without an earring



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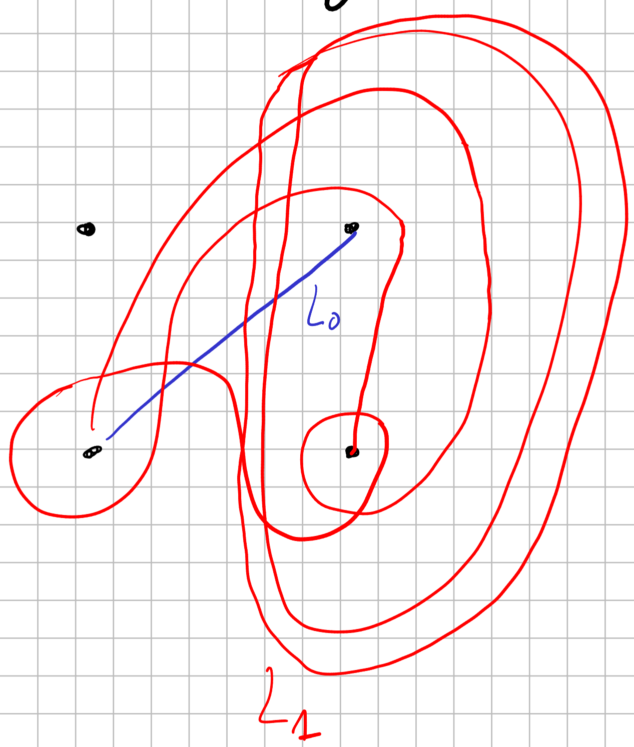
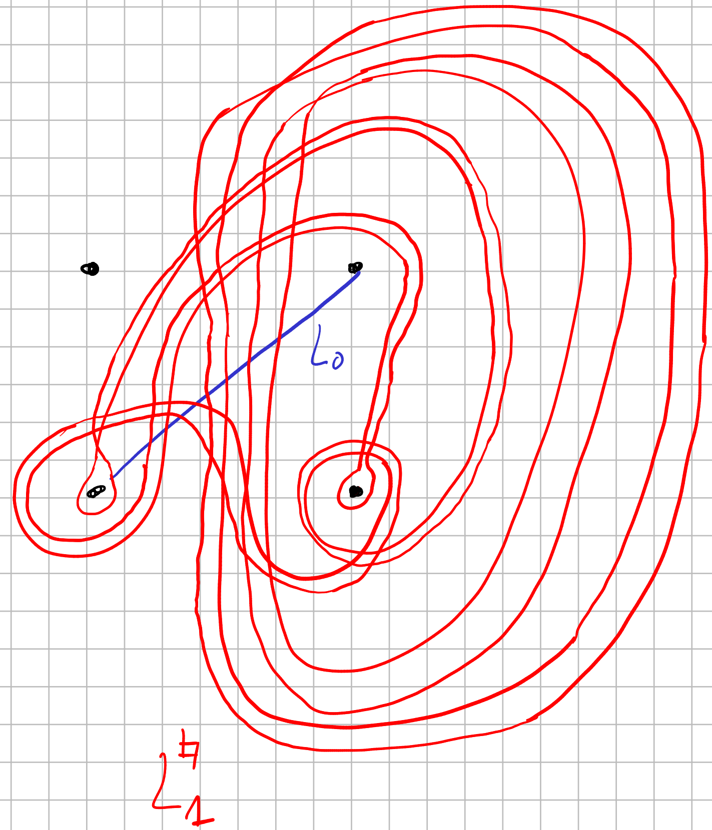
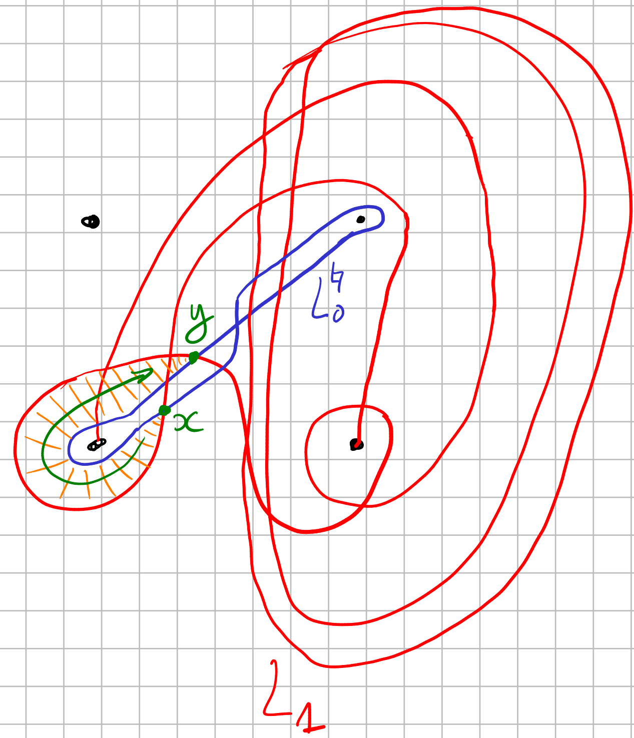


earring on T_1



$\partial = 0$

earring on T_0

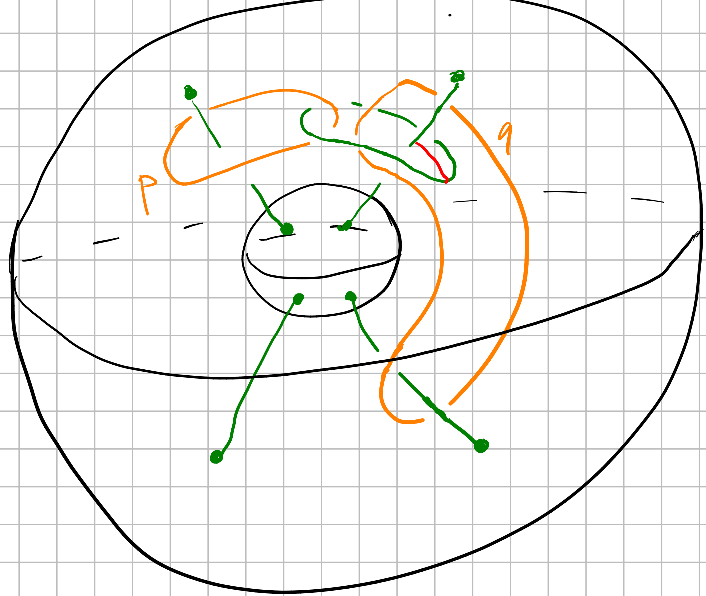


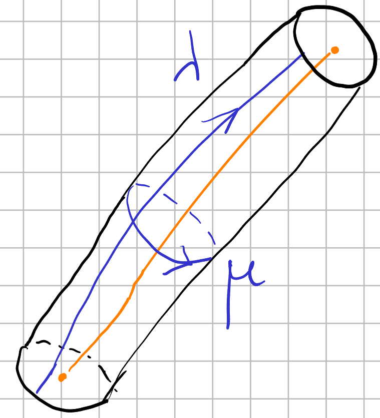
$\partial x = y$

$$HF(L_0, L_1^\natural) \simeq \mathbb{F}^9 \neq HF(L_0^\natural, L_1) \simeq \mathbb{F}^7$$

$$HF(L_0, L_1^\natural) \cong \widetilde{Kh}(mT_{4,5}) \qquad HF(L_0^\natural, L_1) \cong I^\natural(T_{4,5})$$

The earring correspondence

$T =$ 

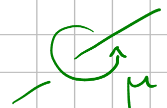


- holonomy perturbations:

$\rho(\mu) = \exp\left(s \cdot \mathrm{Im}(\rho(\lambda))\right)$

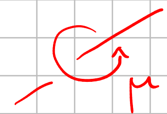
$s \in \mathbb{R}$ small parameter

- traceless holonomy



$\rho(\mu) \in \{\mathrm{Im}\, q = 0\}$

- w_2 condition



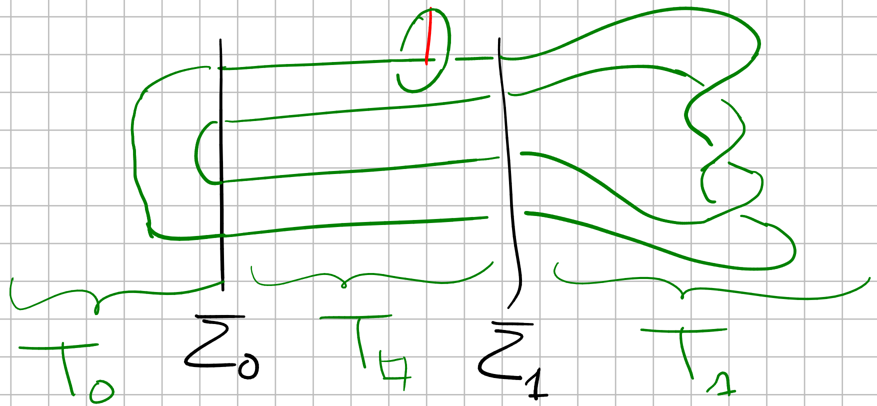
$\rho(\mu) = -1$

$\hookrightarrow \; L_s := \mathcal{L}(T) \looparrowright P_0^- \times P_1$

immersed Lagrangian correspondence

In Weinstein's symplectic category "Symp":

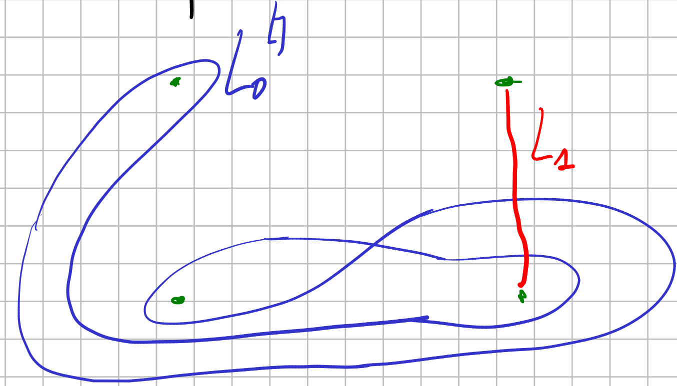
$$P_0 \xrightarrow{L_s} P_1$$



Wehrheim-Woodward:

"Character varieties define a Floer Field theory", i.e. a functor $\mathrm{Tan} \to \mathrm{Symp}$.

$$pt \xrightarrow{L_0} \mathcal{P}_0 \xrightarrow{\Psi_s} \mathcal{P}_1 \xrightarrow{L_1} pt \quad \rightsquigarrow \quad pt \xrightarrow[L_0^\natural = L_0 \circ \Psi_s]{} \mathcal{P}_1 \xrightarrow[L_1]{} pt$$



$$\rightsquigarrow$$

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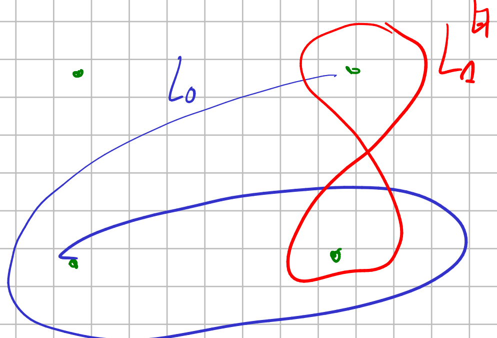
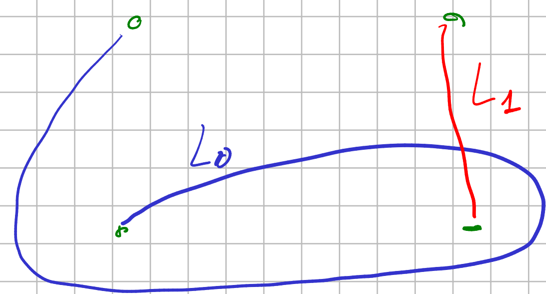
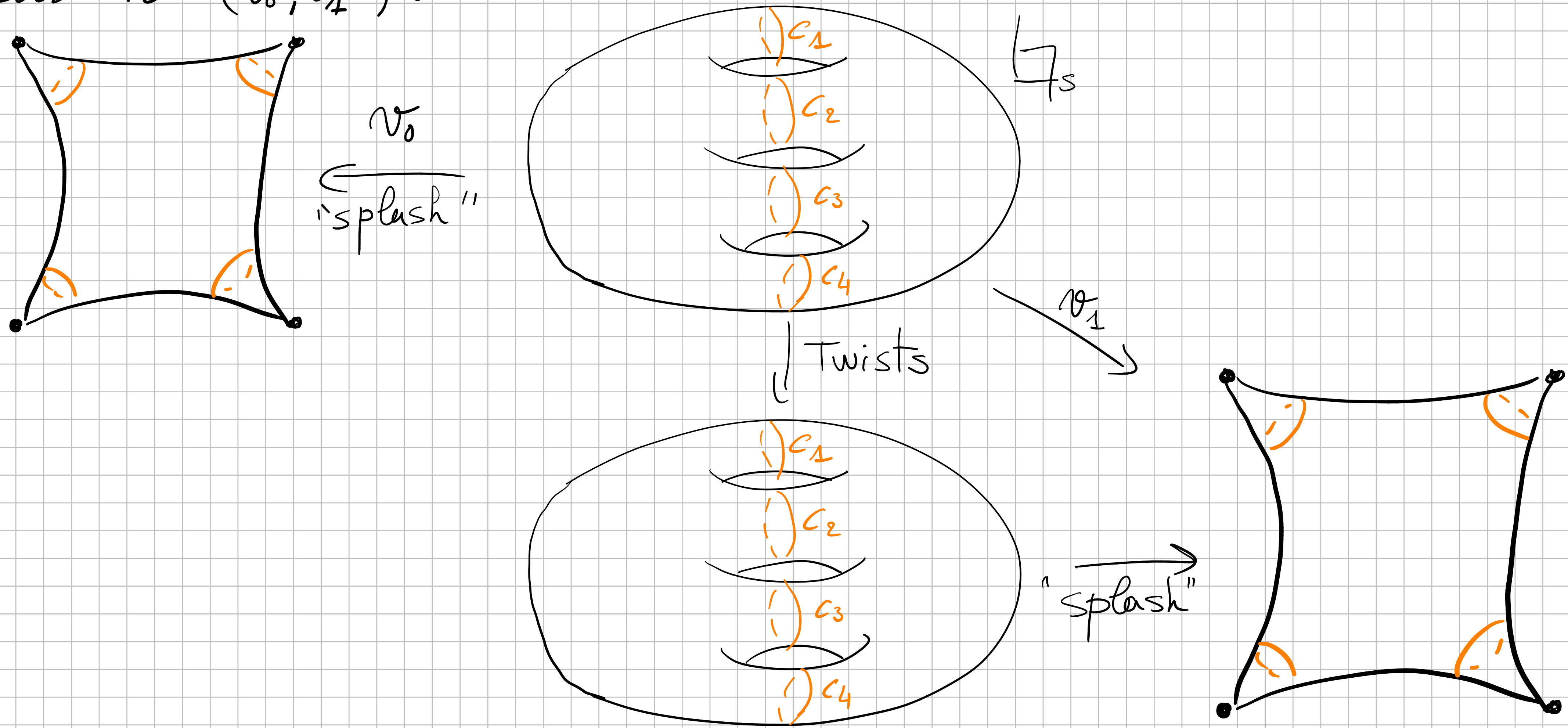




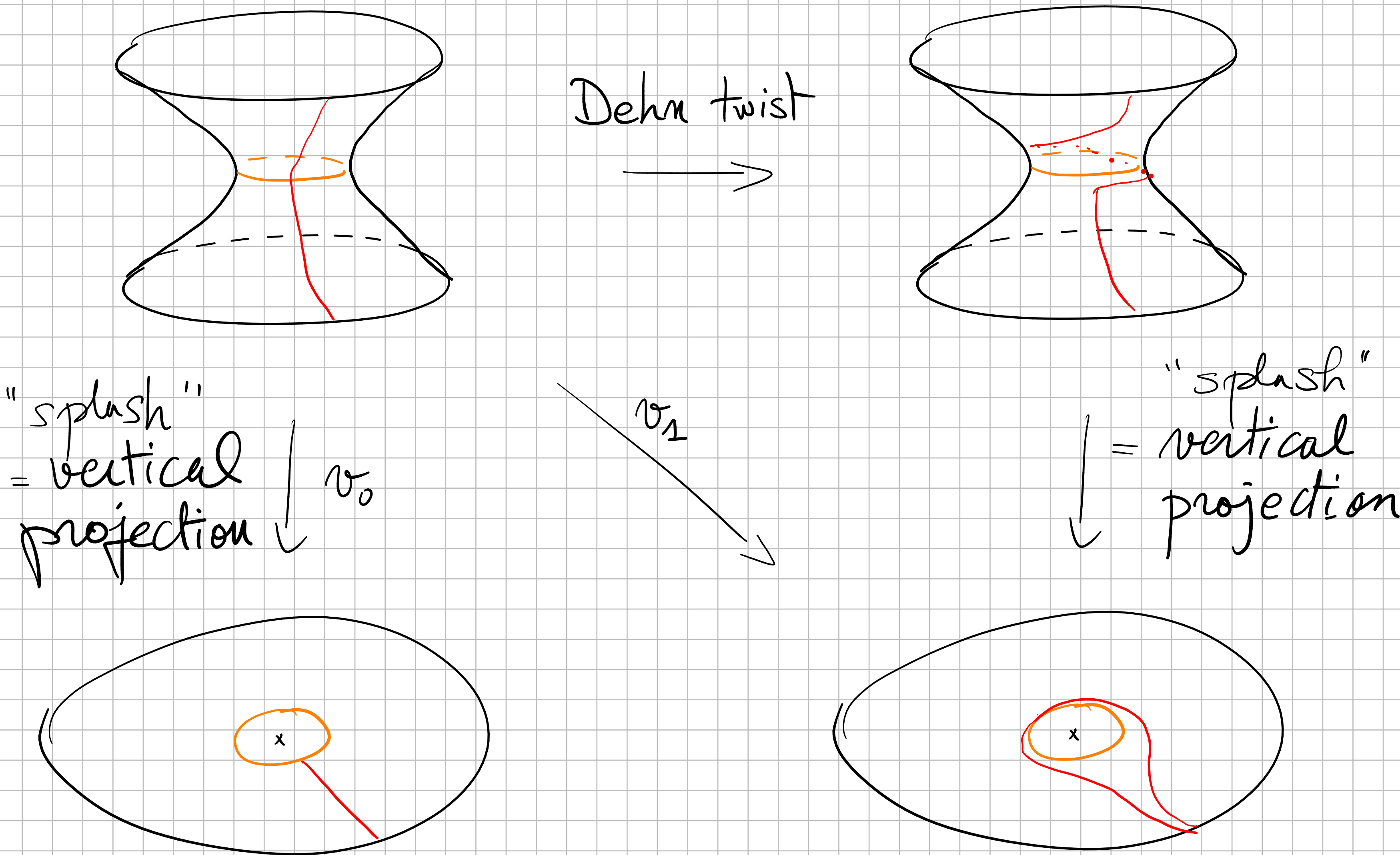
(Without an earring)

Th: [C., Herald, Kink, Kotelskiy] For $s \neq 0$ arb. small,

1. $\Box_s$ is a smooth genus 3 surface.
2. The Lagrangian restriction $\Box_s \xrightarrow{(r_0, r_1)} \bar{P}_0 \times P_1$ avoids the singular stratum $4pt \times P_1 \cup \bar{P}_0 \times 4pt$, and is arbitrarily close to (v_0, v_1):

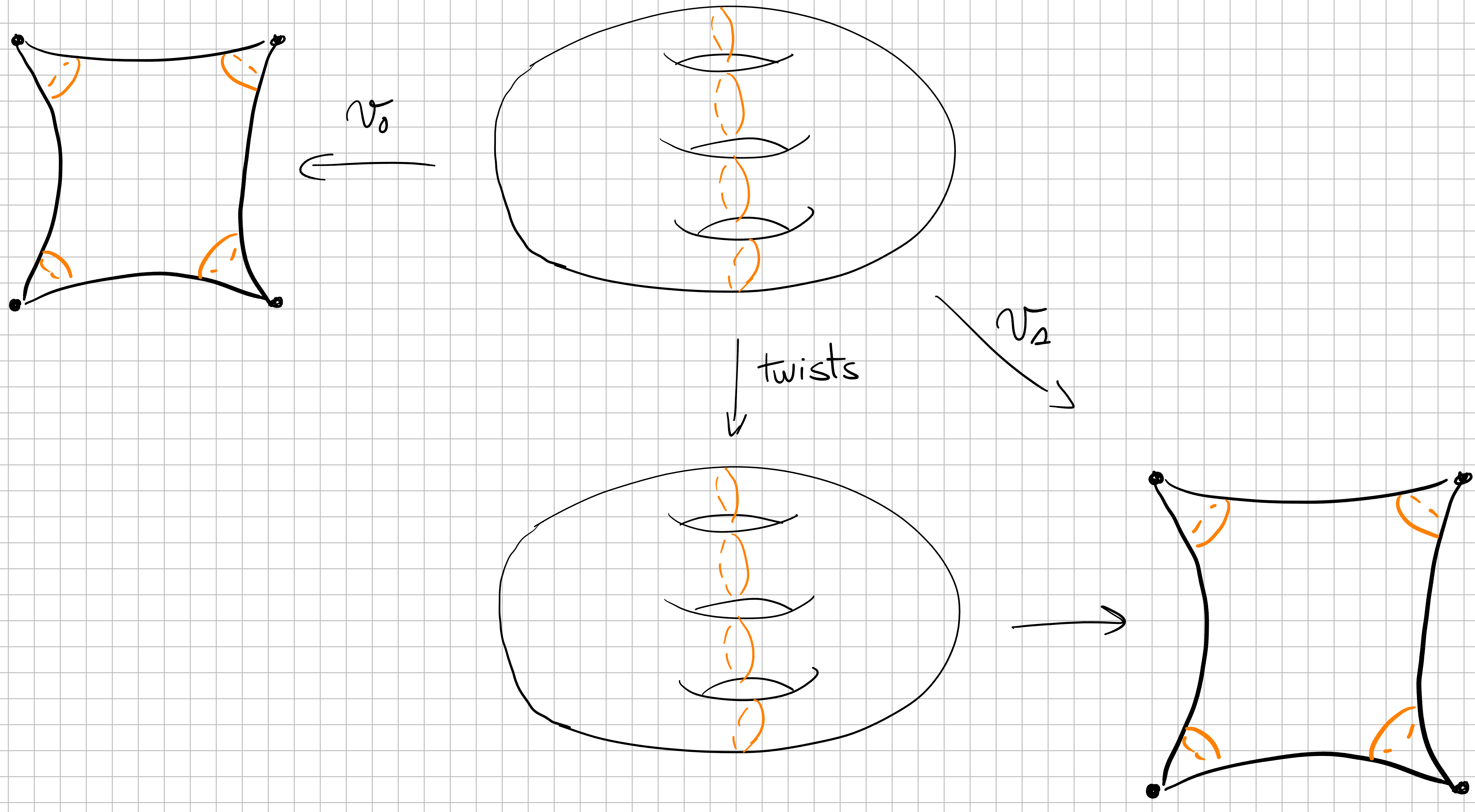


Near the c_i's:


Dehn twist
"splash" = vertical projection
v_0
v_1
"splash" = vertical projection
x
x

3. Action on the Pillowcase curves:

- □ doubles the circles
- □ transforms arcs to "figure eight curves"



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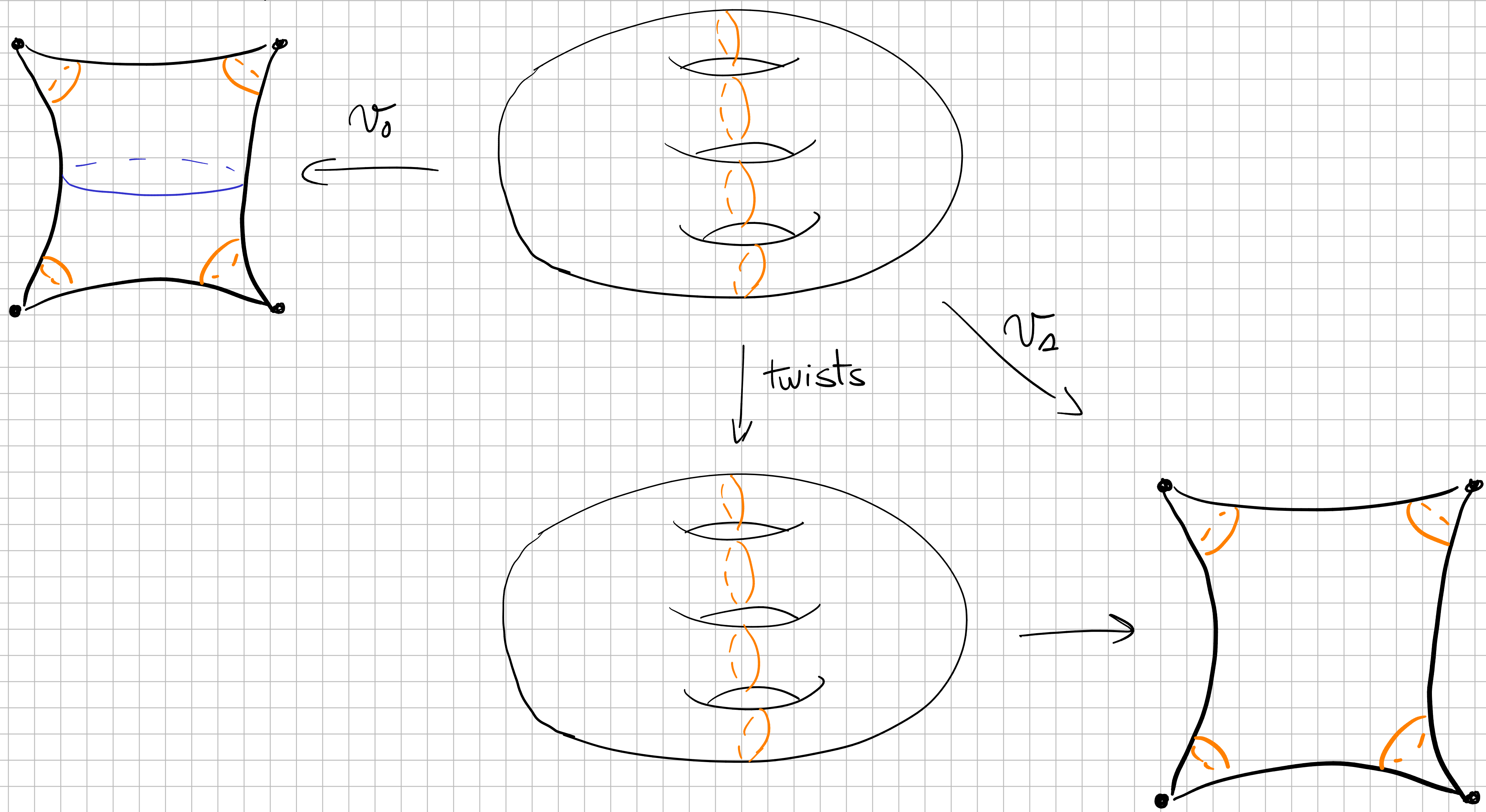

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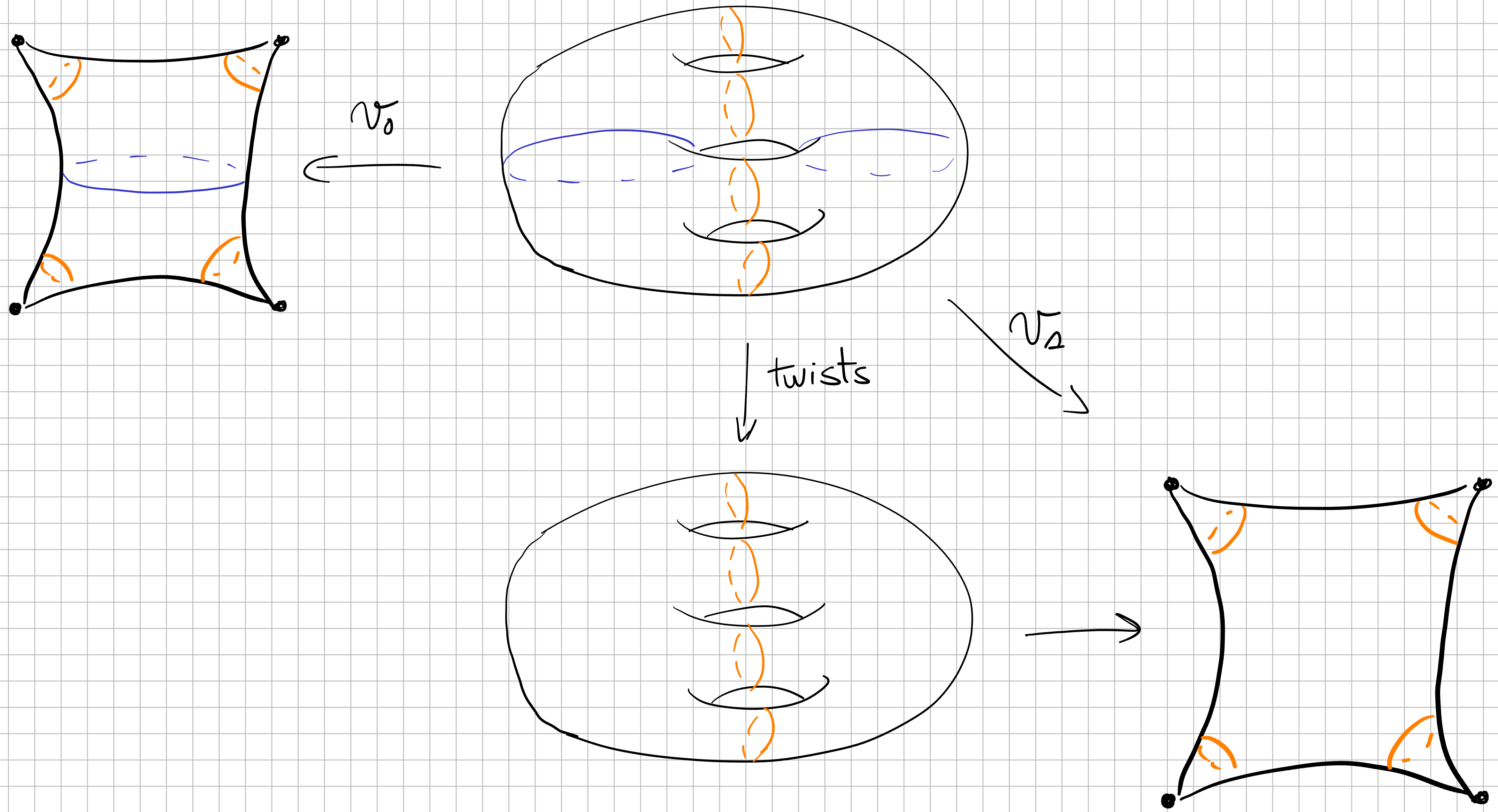

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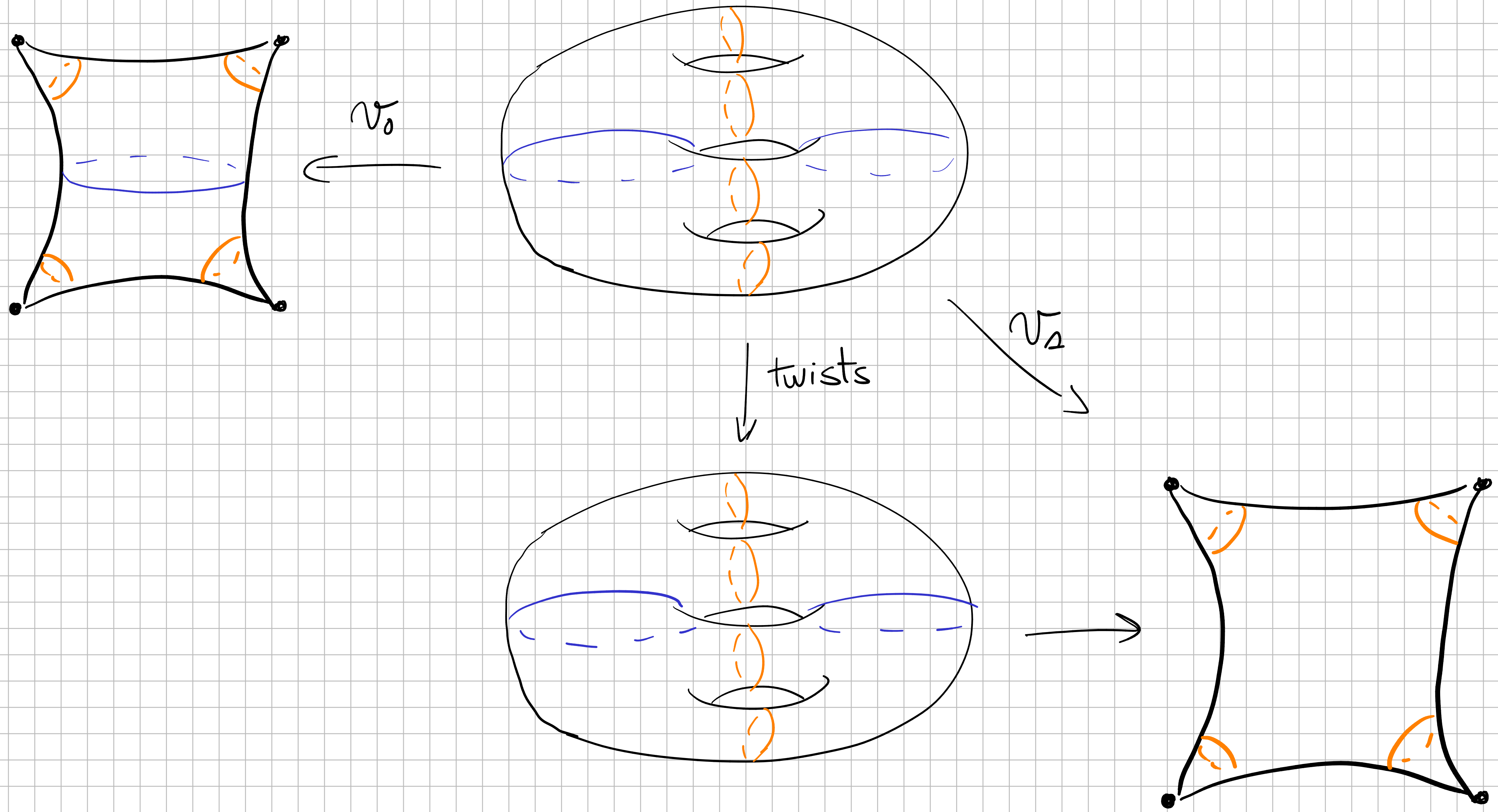

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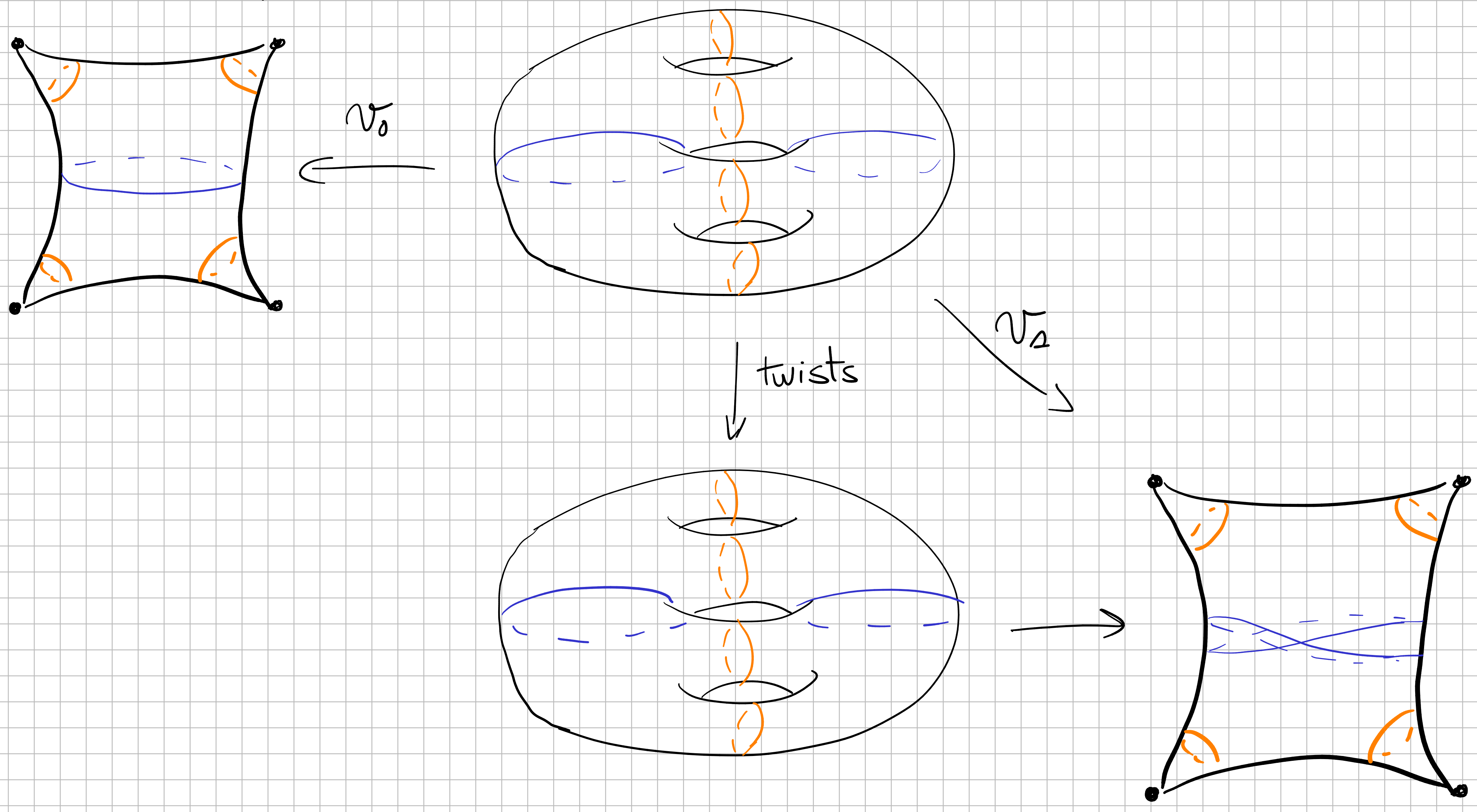

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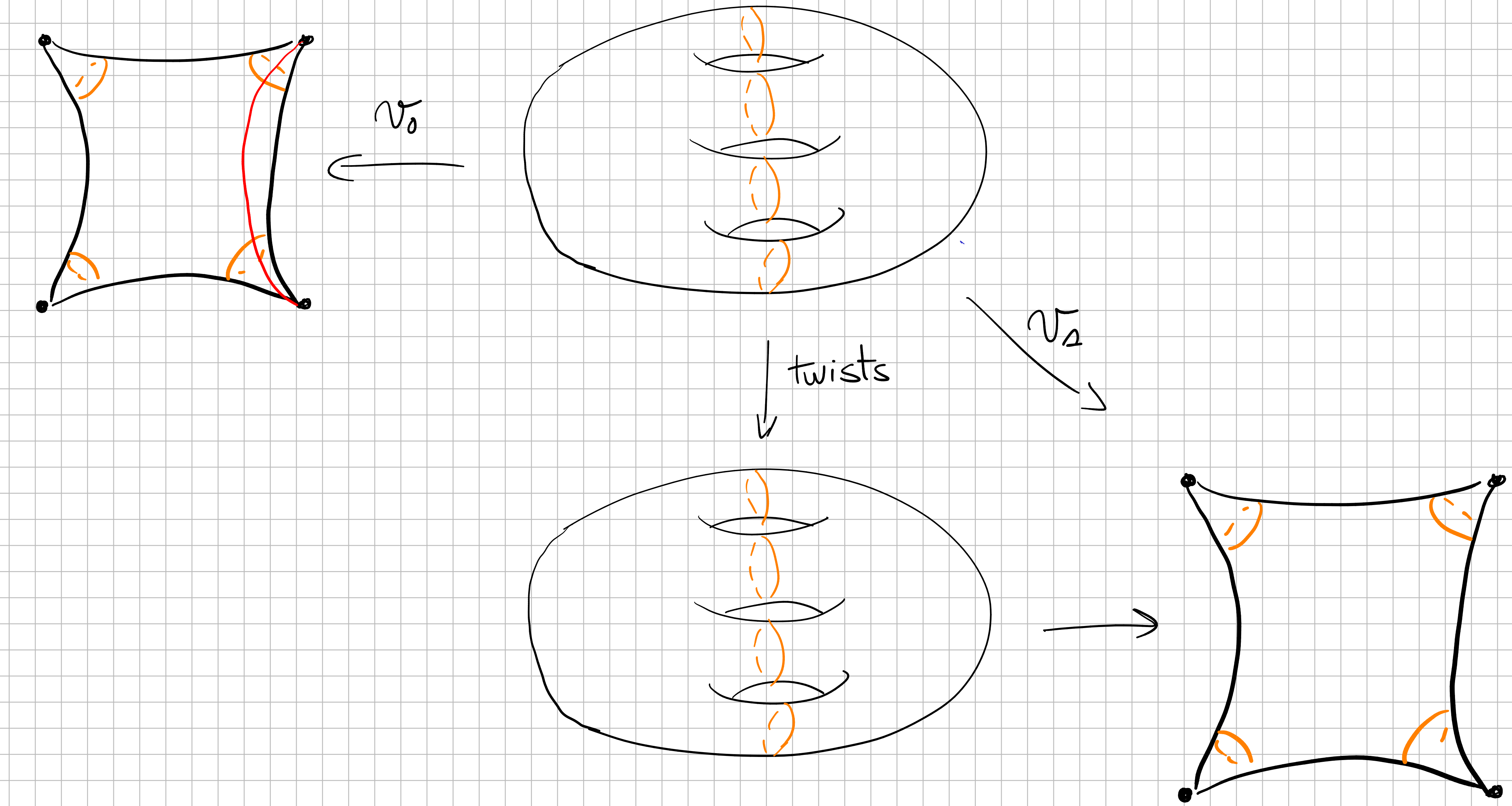

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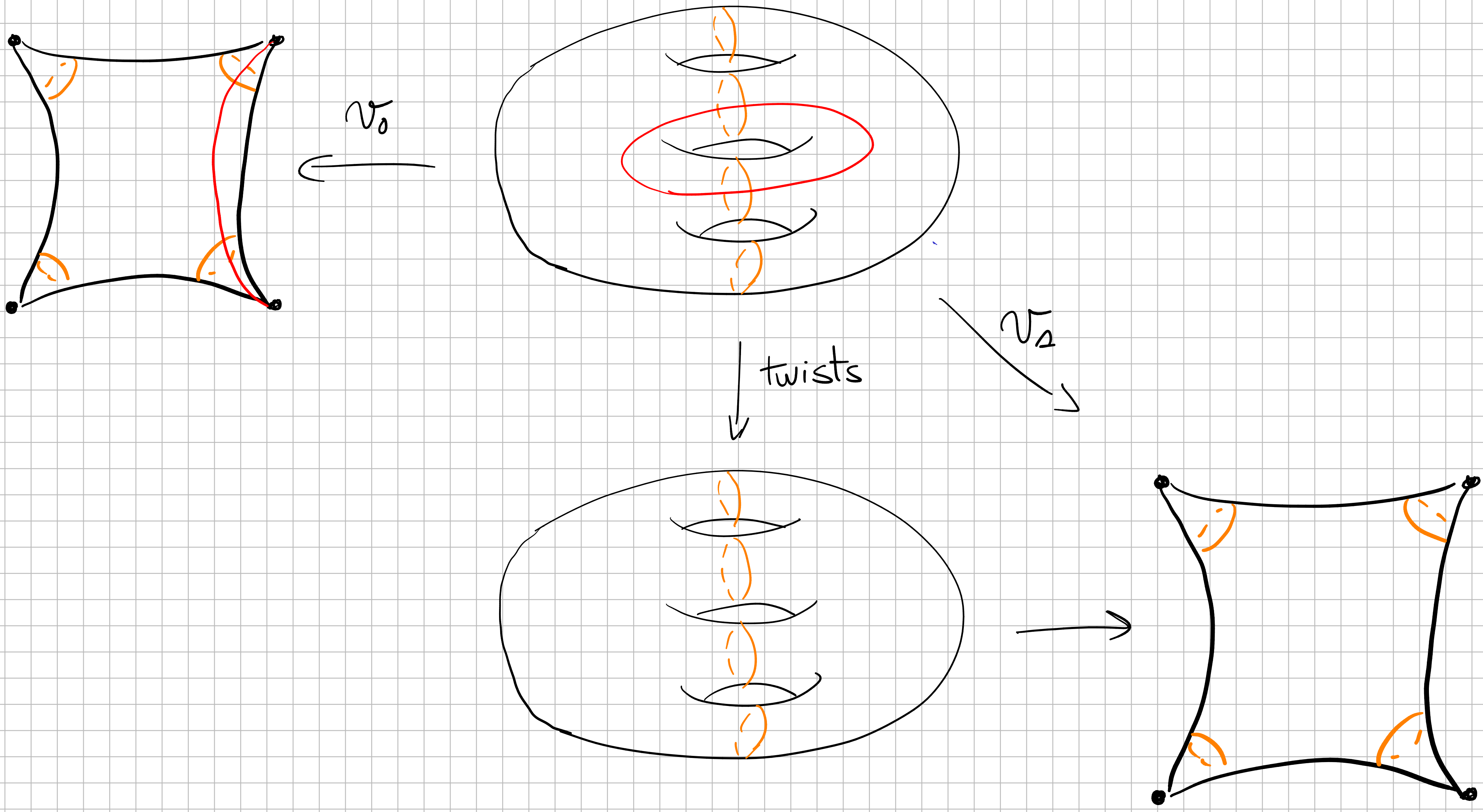

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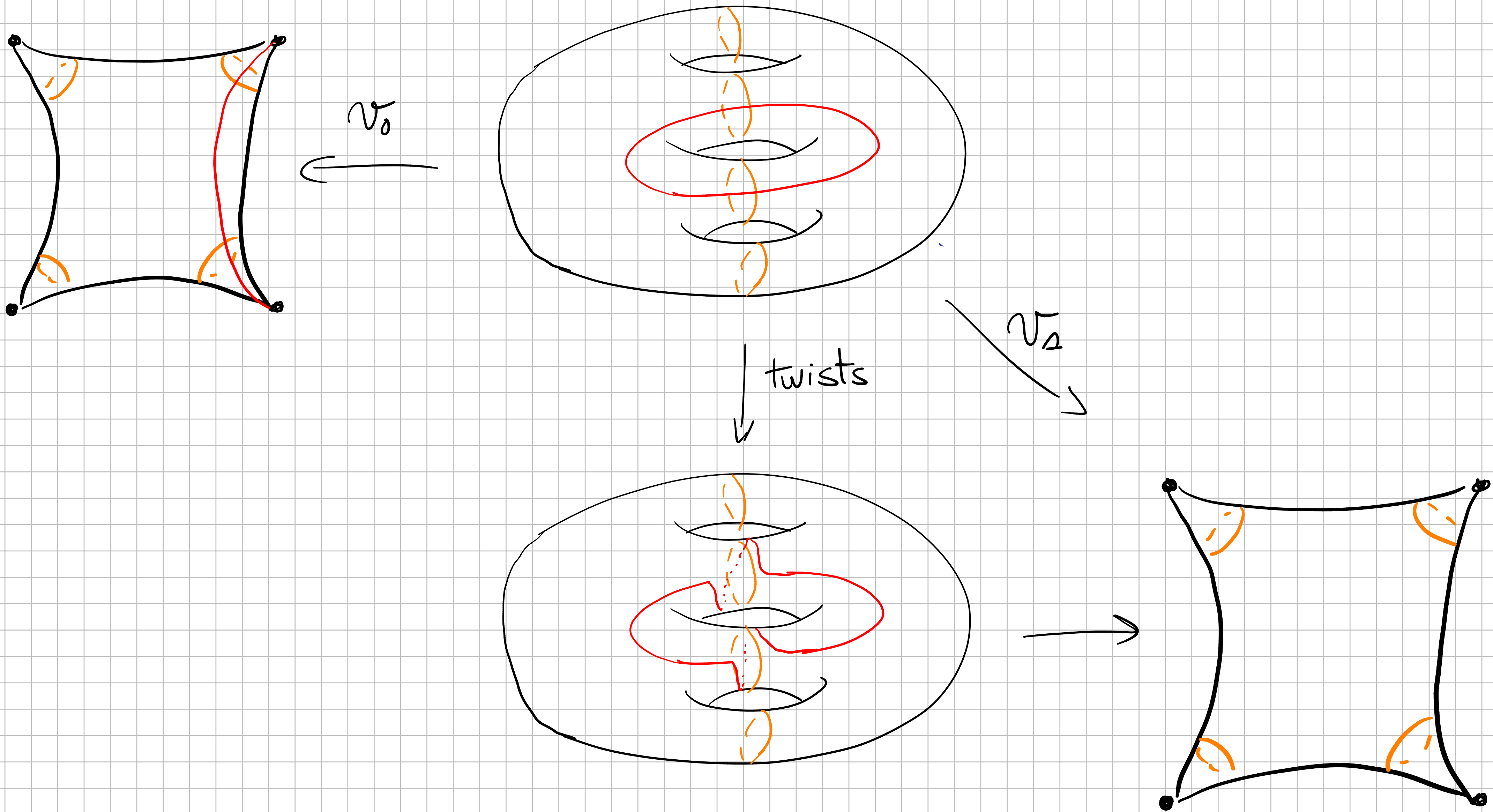

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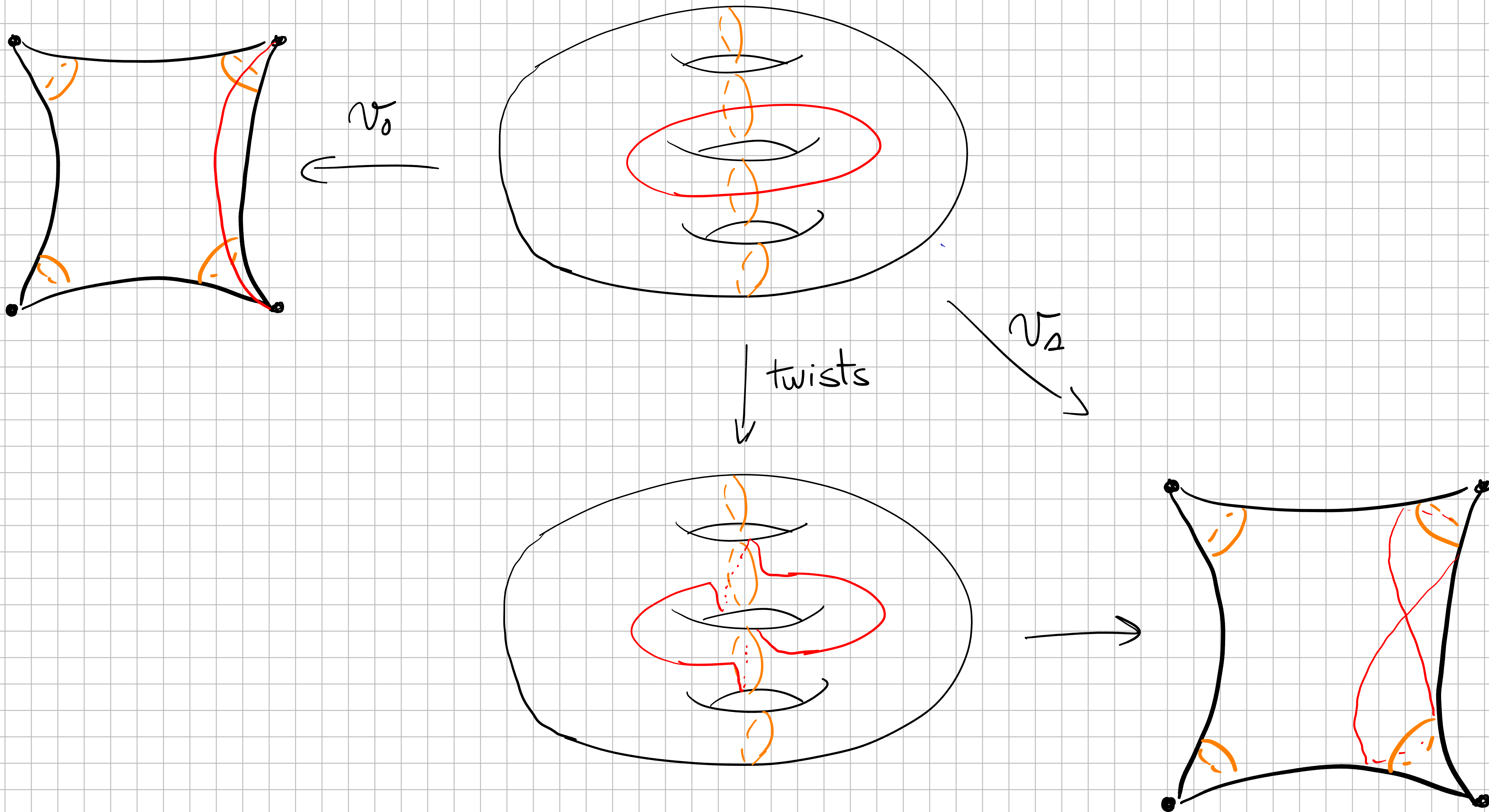

Quilted Floer homology (Wehrheim-Woodward)

$\underline{L} = pt \xrightarrow[L_0]{} M_0 \xrightarrow[L_{01}]{} M_1 \xrightarrow[L_{12}]{} M_2 \xrightarrow[L_2]{} pt$: sequence of Lagrangian corresp.

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$\rightarrow CF(\underline{L}) = \bigoplus_{\underline{x} \in \mathcal{I}(\underline{L})} \mathbb{F} \cdot \underline{x}$,

$$\mathcal{I}(\underline{L}) = \left\{ \underline{x} = (x_0, x_1, x_2) \;\middle|\; \begin{array}{l} x_0 \in L_0 \\ (x_0, x_1) \in L_{01} \\ (x_1, x_2) \in L_{12} \\ x_2 \in L_2 \end{array} \right\}$$

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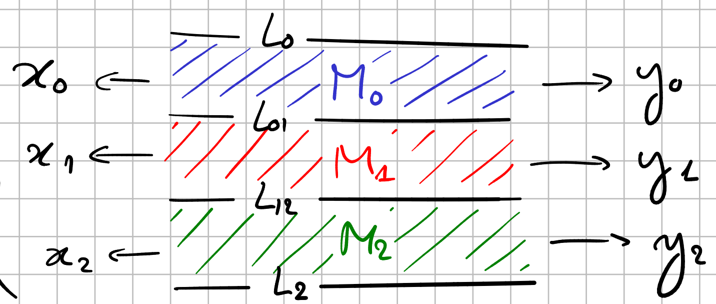
$\partial : CF(\underline{L}) \to CF(\underline{L})$ counts pseudo-holomorphic "quilts" :

$\langle \partial \underline{x}, \underline{y} \rangle = \#$ {



} $u_i : \mathbb{R} \times [i, i+1] \to M_i$

$\left(u_i(s, i+1), u_{i+1}(s, i+1) \right) \in L_{i(i+1)}$

Composition of correspondences

$$M_0 \xrightarrow{L_{01}} M_1 \xrightarrow{L_{12}} M_2$$

$$L_{02} = L_{01} \circ L_{12} := \left\{ (x_0, x_2) \;\middle|\; \exists x_1 : \begin{matrix} (x_0, x_1) \in L_{01} \\ (x_1, x_2) \in L_{12} \end{matrix} \right\} = \pi_{02}\left(L_{01} \times M_2 \cap M_0 \times L_{12} \right)$$

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- transversality
- embedding

<u>Th</u> [Wehrheim – Woodward]

If the composition of L_{01} and L_{12} is embedded* (+ other assumptions),

then $$HF(L_0, L_{01}, L_{12}, L_2) \simeq HF(L_0, L_{01} \circ L_{12}, L_2)$$

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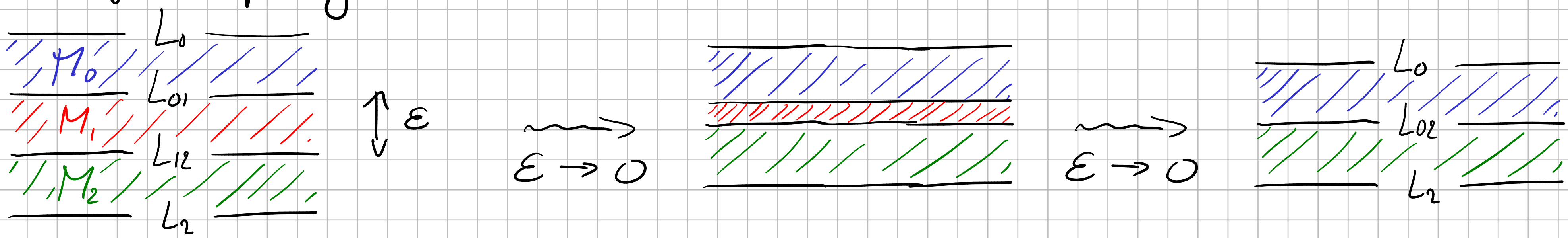
Idea of the proof:



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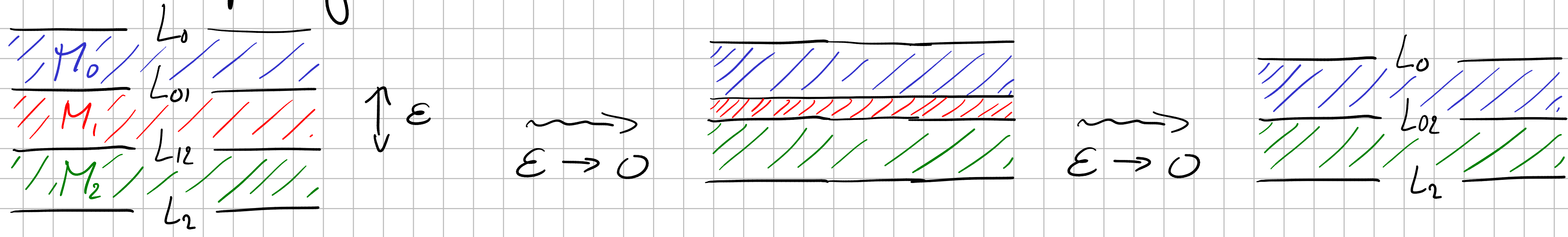
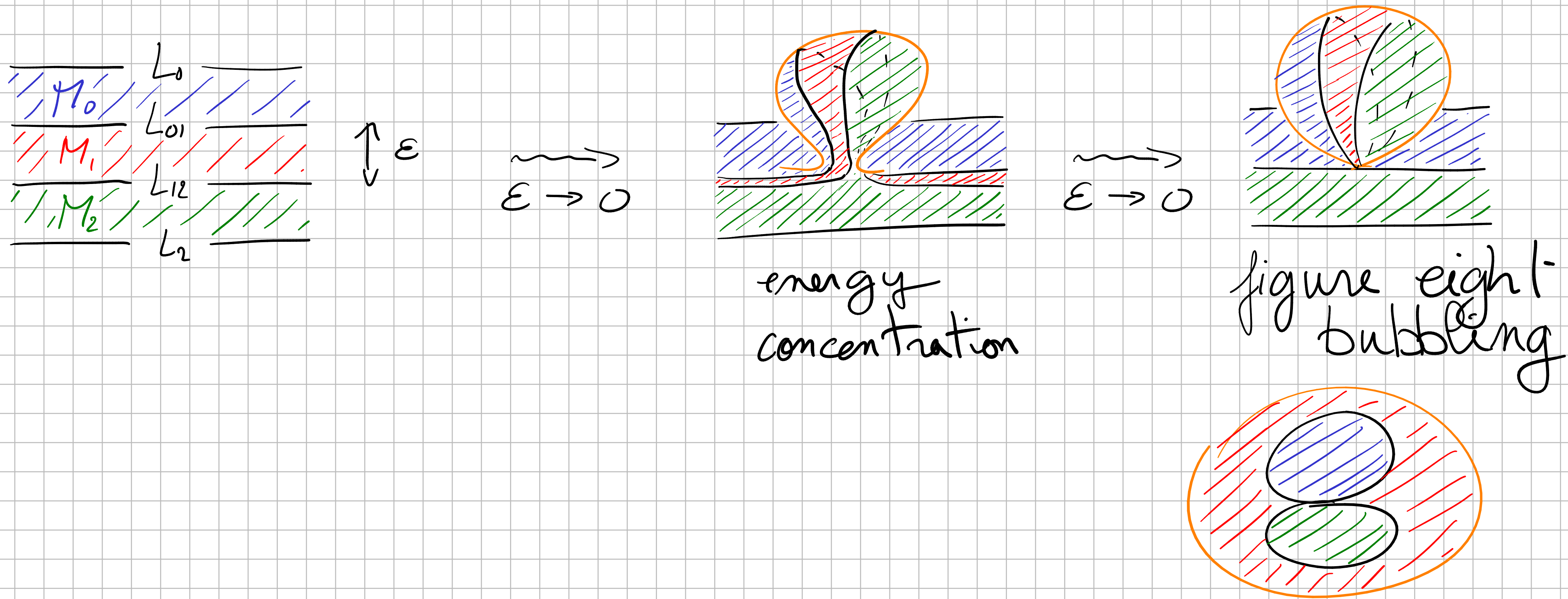
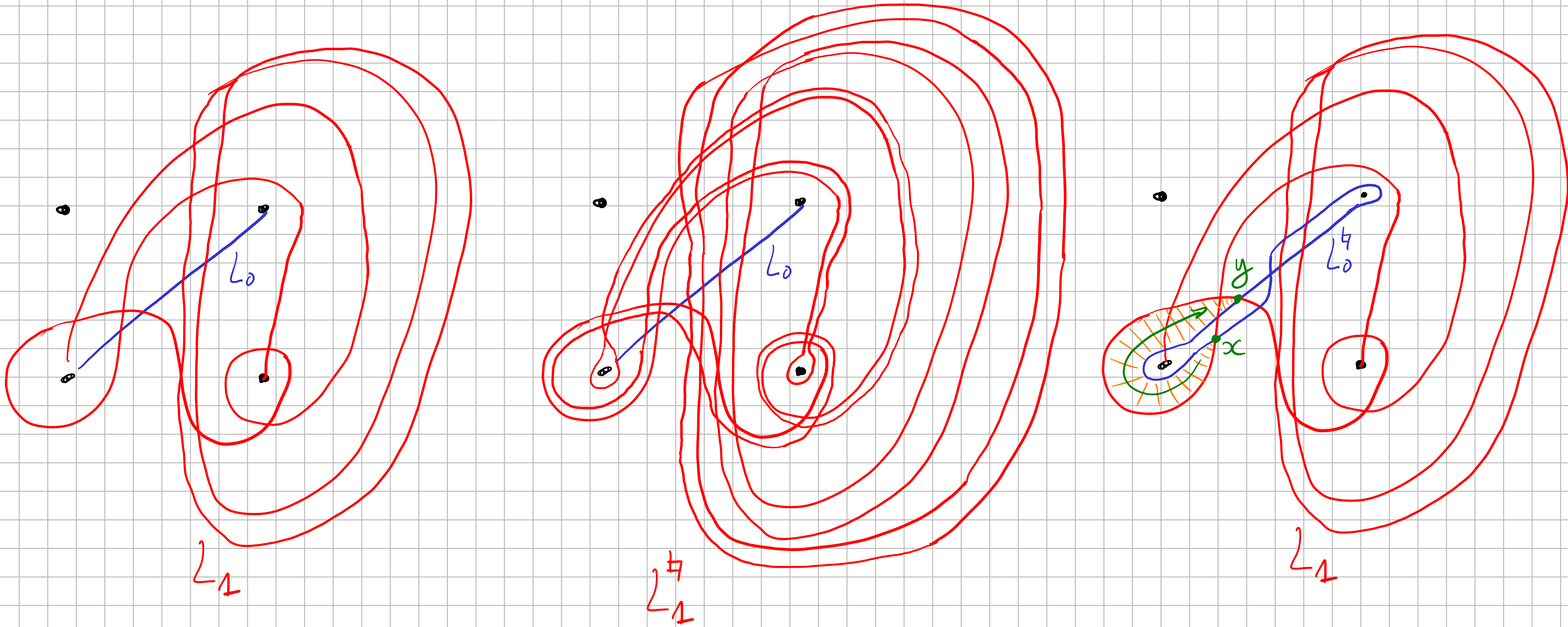


Can happen if composition is not embedded:



→ Back to $T_{4,5}$



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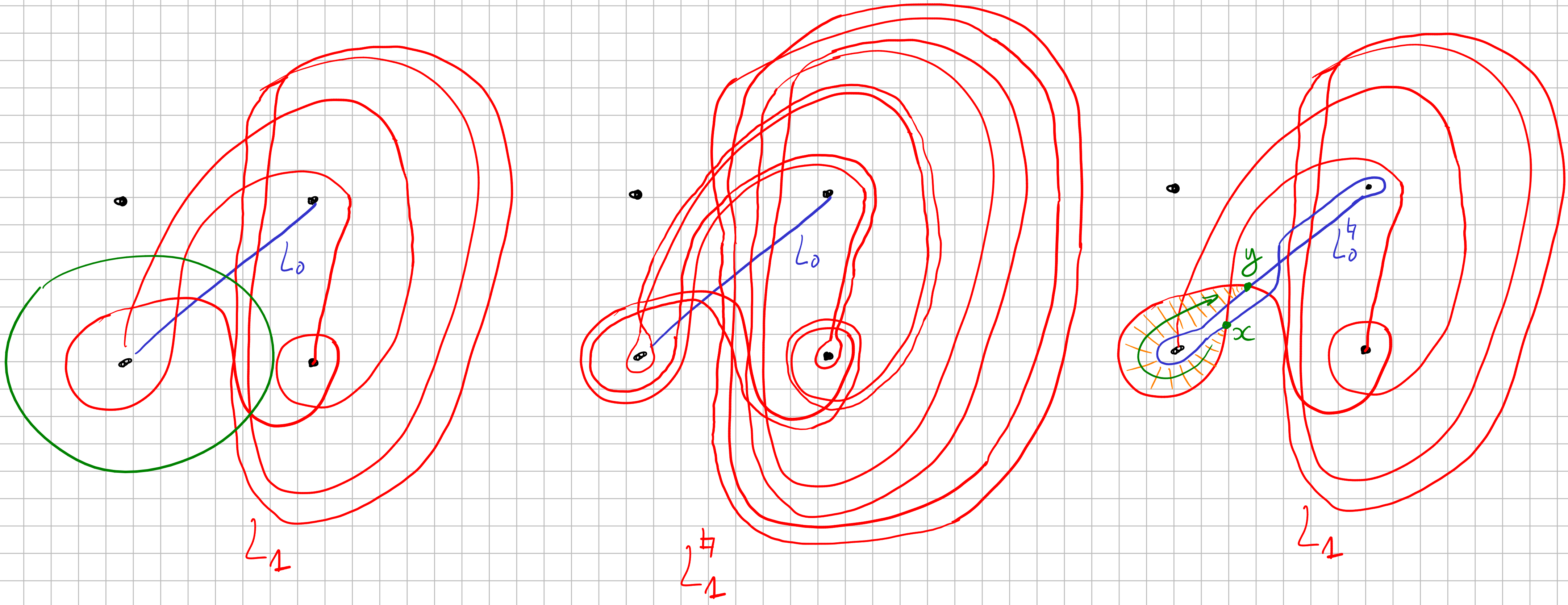
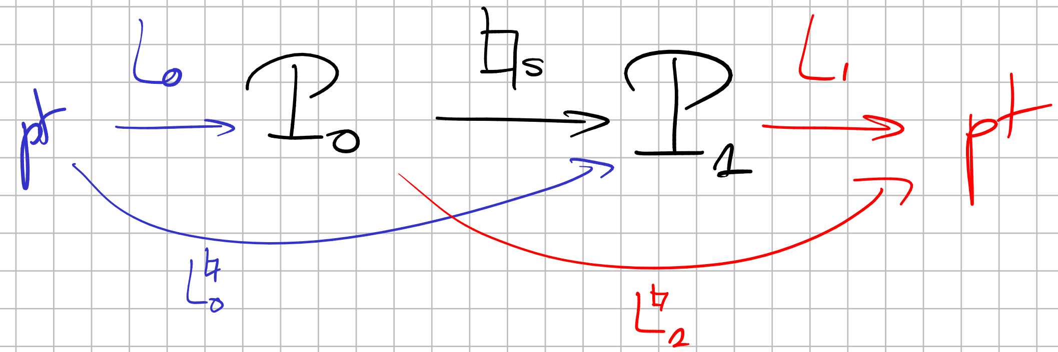
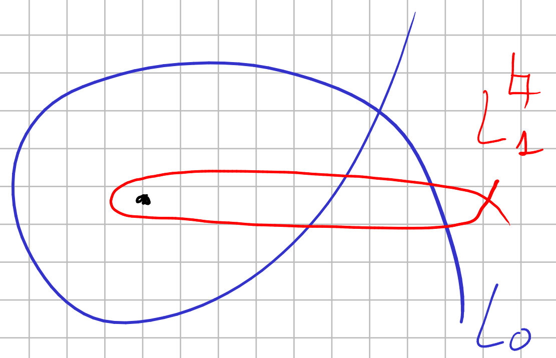
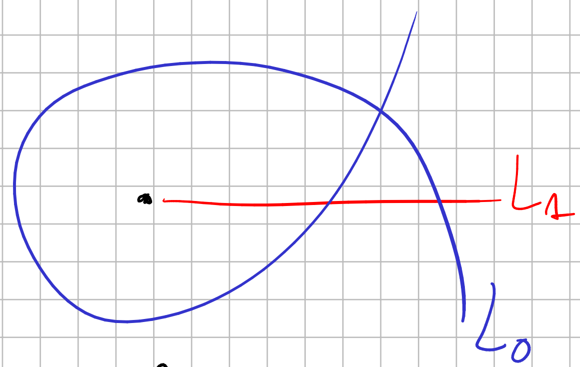


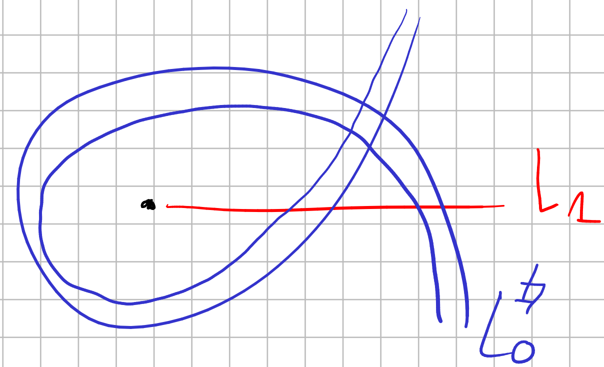




In P_0

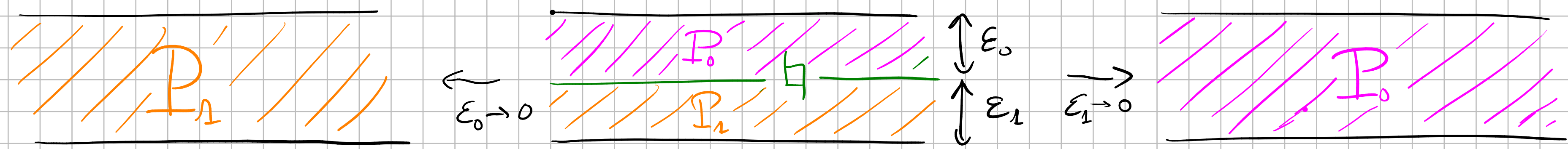


Without an earring

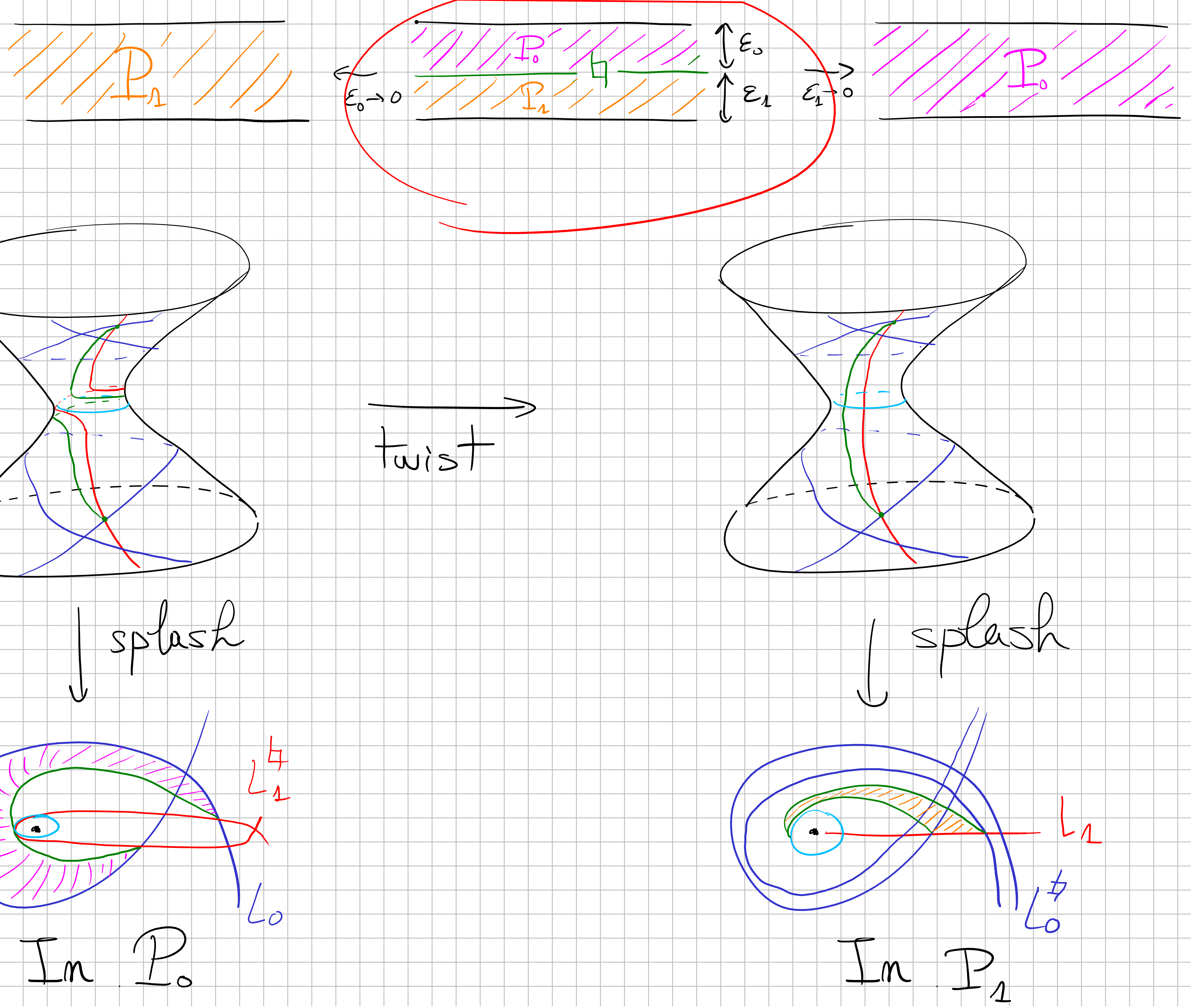


In P_1

Idea:

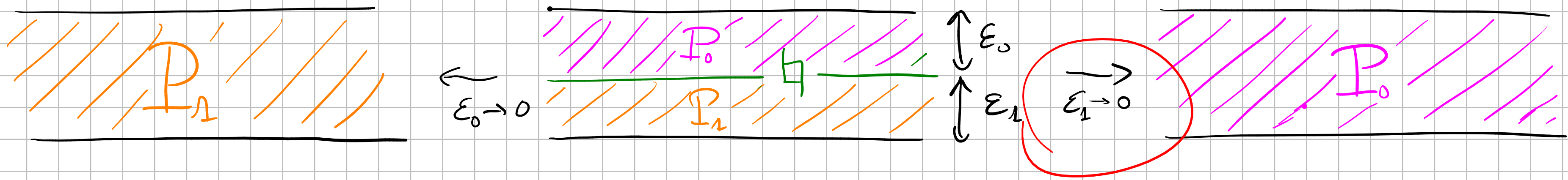


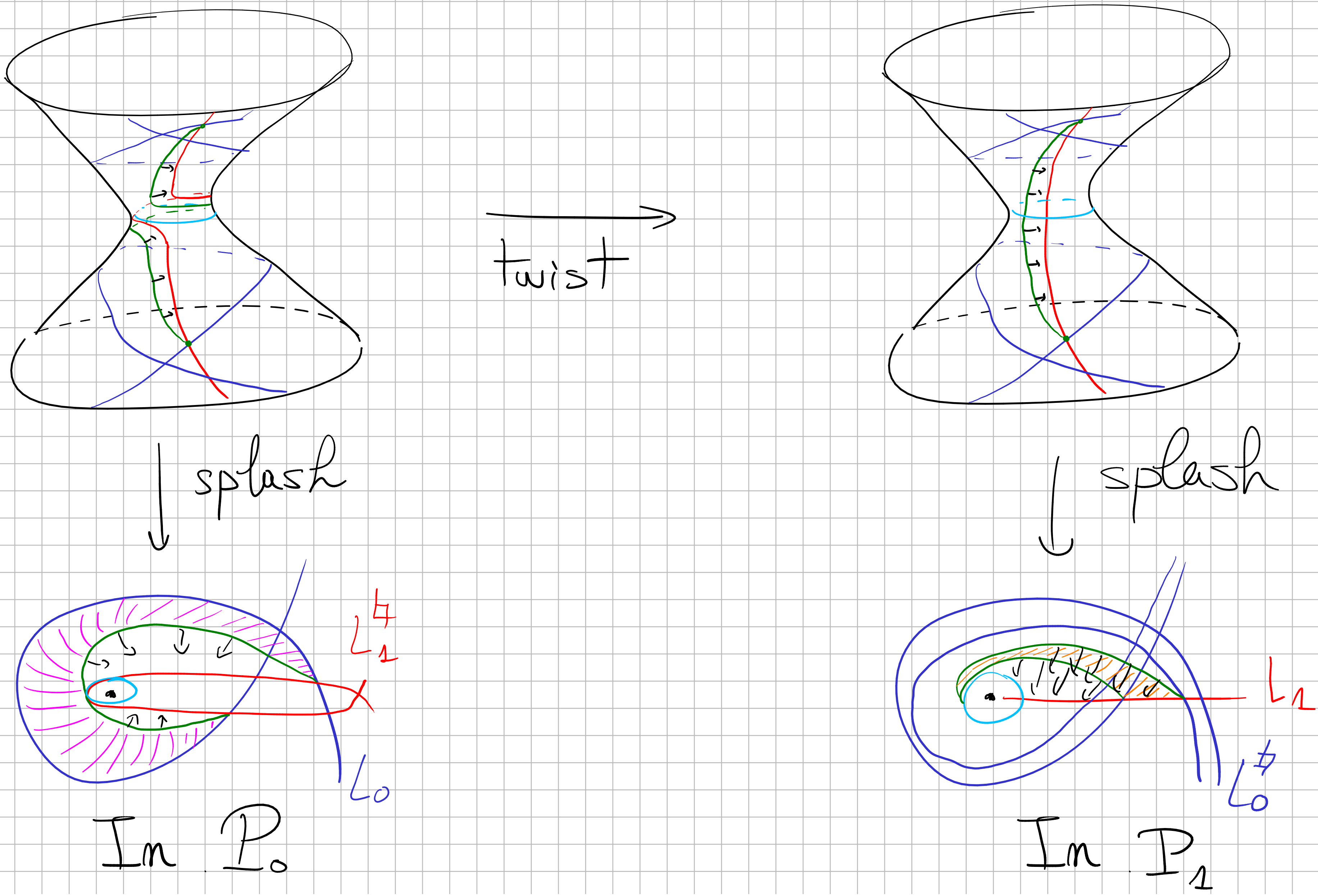
Idea:



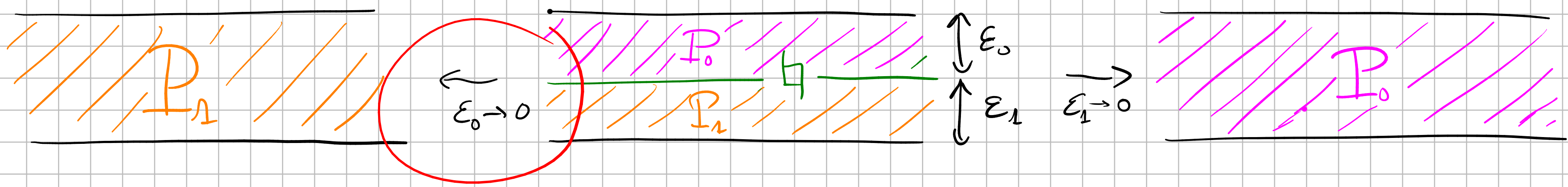
P1
P0
P1
ε0
ε1
ε0 → 0
ε1 → 0
P0
twist
splash
splash
L1
L0
In P0
L1
L0
In P1

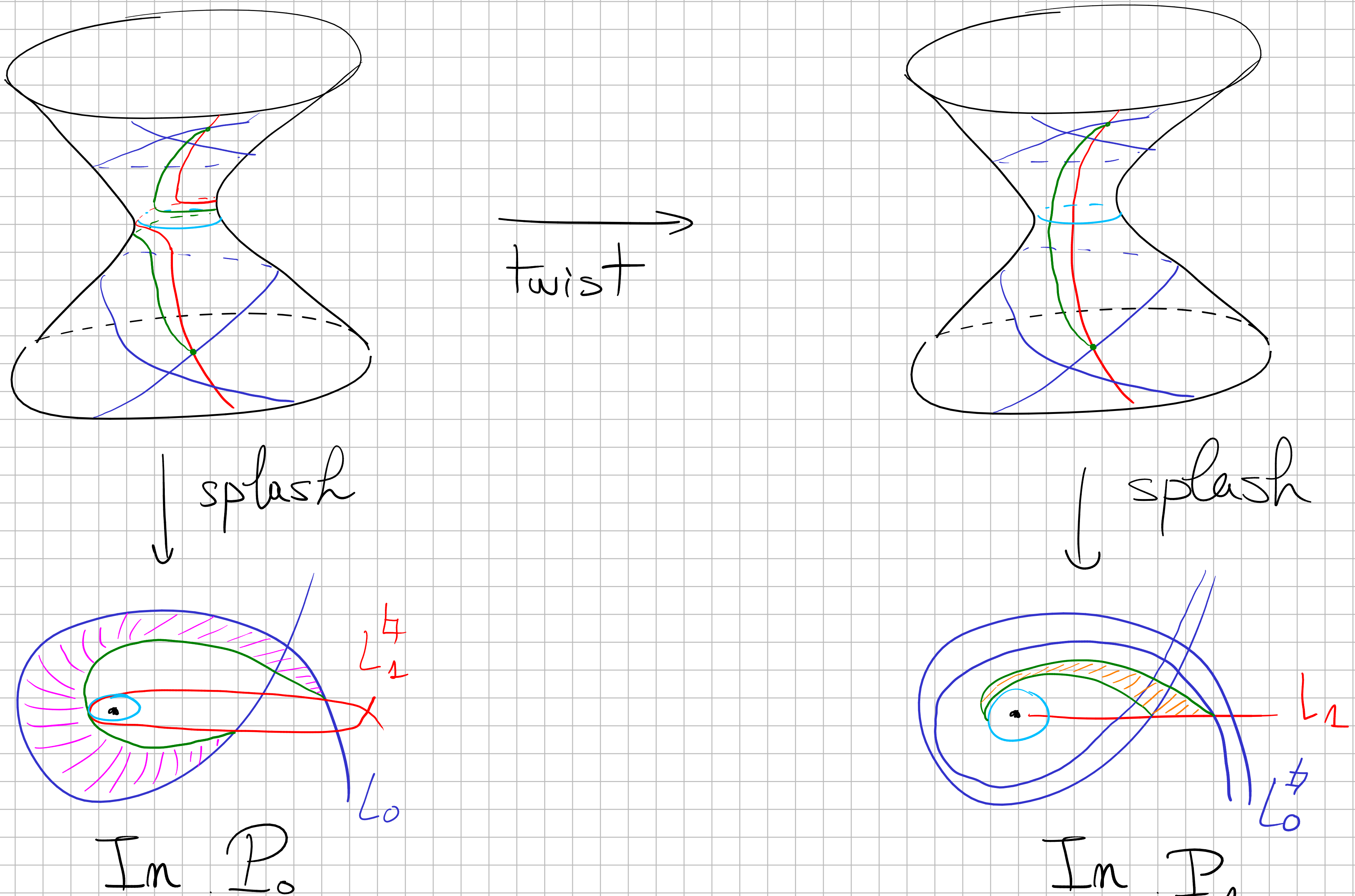
Idea:


P1
ε0 → 0
P0
P1
ε0
ε1
ε1 → 0
P0

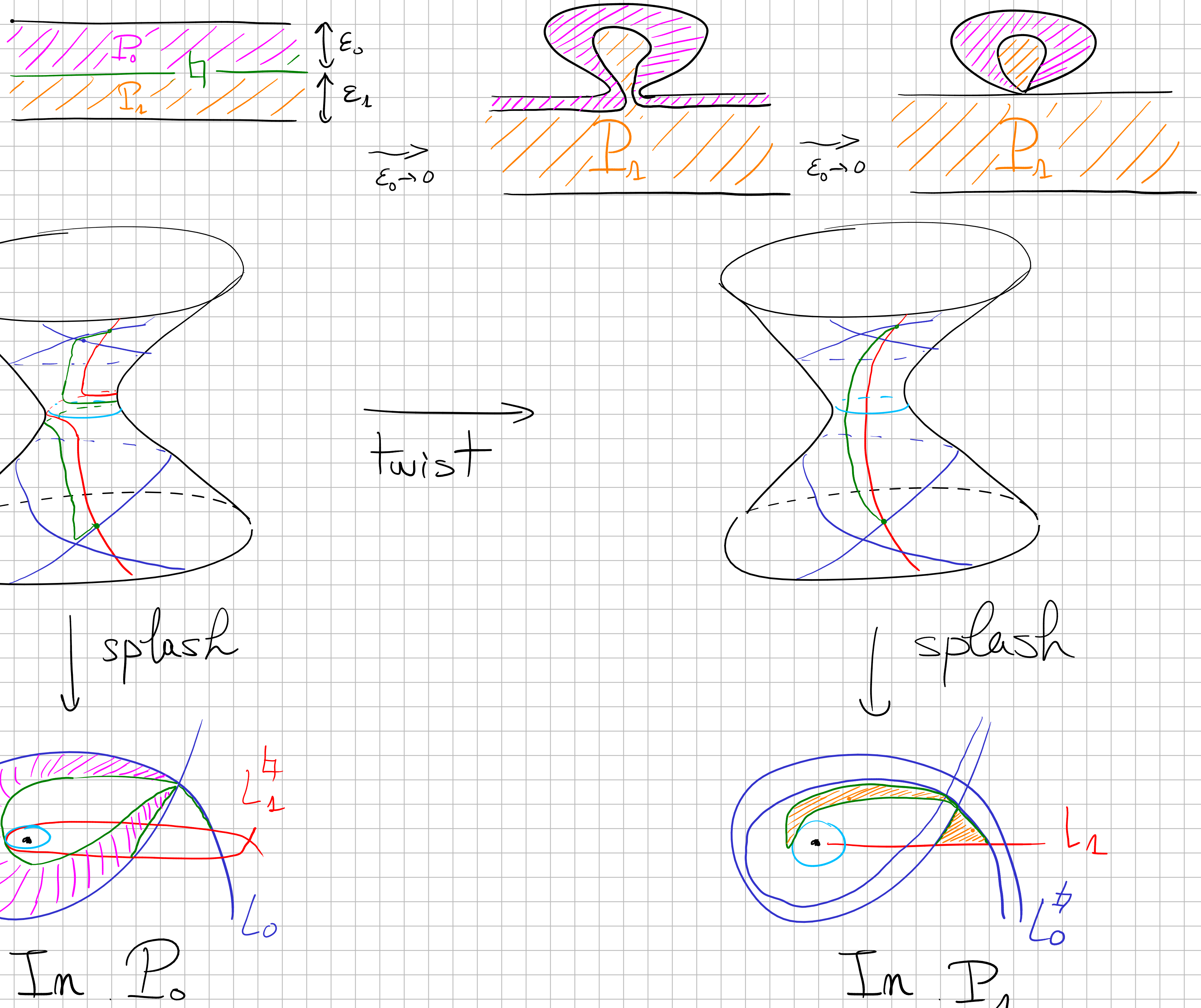

twist
splash
splash
L1
L0
L1
L0
In P0
In P1

Idea:


P1
ε0 → 0
P0
P1
ε0
ε1
ε1 → 0
P0

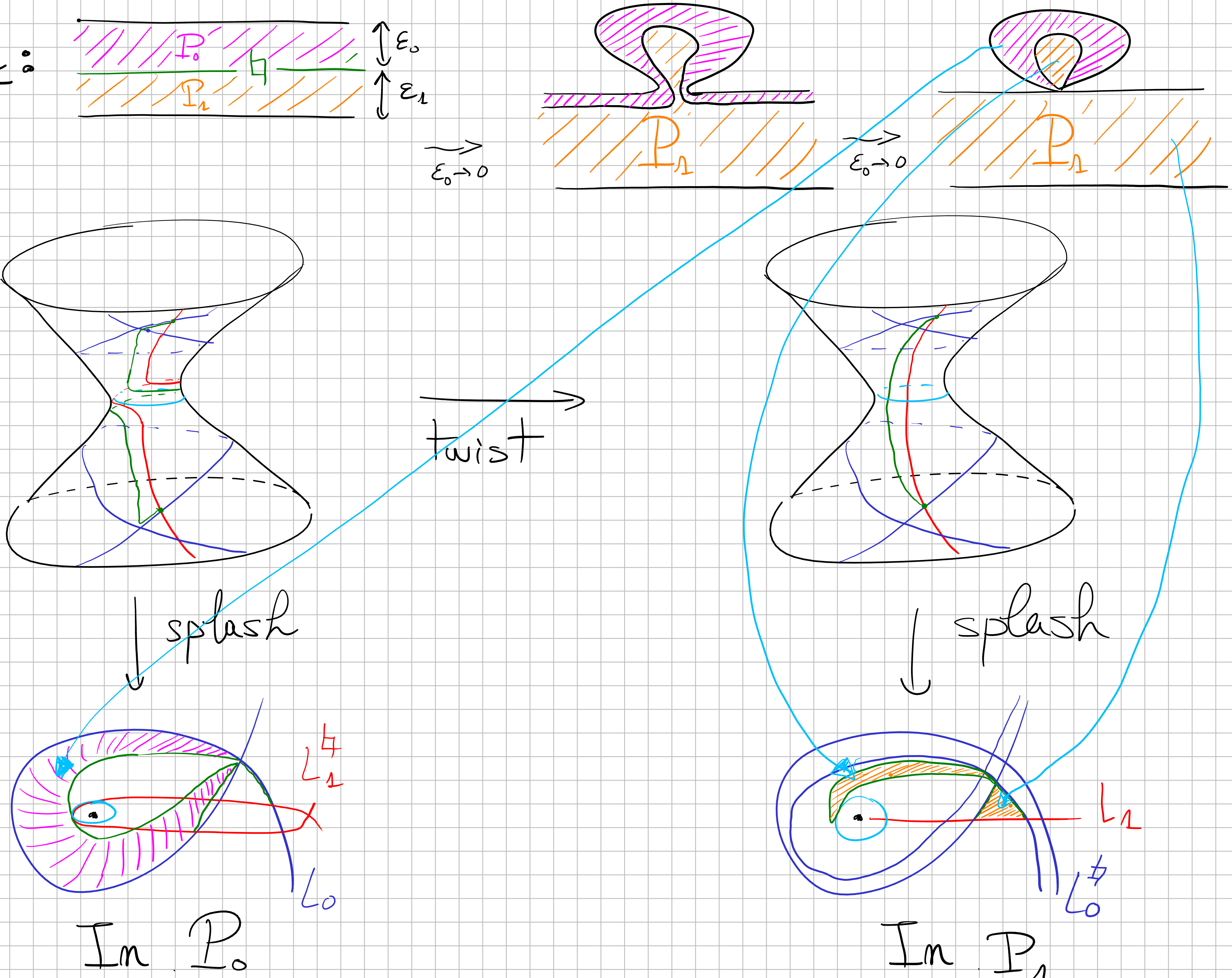

twist
splash
splash
L1
L0
L1
L0
In P0
In P1

Idea:



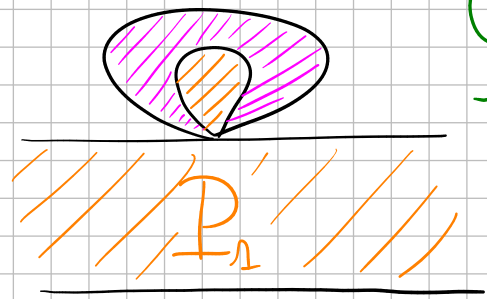
P_0
P_1
ε_0
ε_1
$\varepsilon_0 \to 0$
P_1
$\varepsilon_0 \to 0$
P_1
twist
splash
splash
$L_1^\natural$
L_0
In P_0
L_1
$L_0^\natural$
In P_1

Idea:



P0
P1
ε0
ε1
ε0 → 0
P1
ε0 → 0
P1
twist
splash
splash
L1
L0
L1
L0
In P0
In P1

Algebraic Interpretation



} Bounding Cochain

} μ^2 operation in the Fukaya category

- $(A, \mu^0, \mu^1, \mu^2, \dots)$ A_∞-Algebra : $\mu^k : A^{\otimes k} \longrightarrow A$
- $b \in A$ "bounding cochain" : $\mu^0 + \mu^1(b) + \mu^2(b,b) + \dots = 0$

$\rightsquigarrow \partial^b = \mu^1 + \mu^2(-,b) + \mu^2(b,-) + \mu^3(-,b,b) + \mu^3(b,-,b) + \mu^3(b,b,-) + \dots$

new differential

- Bottman-Wehrheim's Conjecture

\+ • "Floer Field theory" (Wehrheim-Woodward) applied to traceless char. varieties

(conj.) $\Longrightarrow$ Procedure for producing bounding cochains and rectifying the def. of Pillowcase homol.