Relative genus bounds in indefinite 4-manifolds

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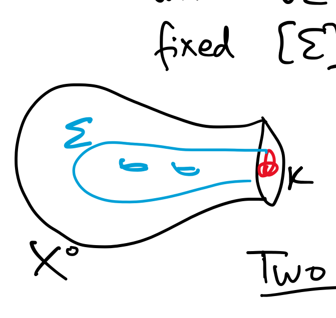
joint work with Marco Marengon and Lisa Piccirillo

X^4 = closed, oriented, smooth $X^\circ = X - \mathring{B}^4$, $K \subset \partial X^\circ = S^3$

Problem: Find lower bounds on $g(\Sigma)$, $\Sigma \hookrightarrow X^\circ$ compact oriented, smoothly, properly embedded, with $\partial\Sigma = K$, with fixed $[\Sigma] \in H_2(X^\circ, \partial X^\circ) = H_2(X)$.



Two well-studied cases:

① K = unknot, $\hat{\Sigma} = \Sigma \cup D^2 \subset X = X^\circ \cup B^4$ $g(\hat{\Sigma}) = g(\Sigma)$

⇝ minimal genus problem in closed 4-manifolds

gauge theory ⇝ sample results:

Thom Conjecture (Kronheimer-Mrowka 1994):
$\min\{g(\Sigma) \mid \Sigma \subset \mathbb{CP}^2, [\Sigma] = d[\mathbb{CP}^1]\} = \frac{(d-1)(d-2)}{2}$

Symplectic Thom Conj. (Ozsváth-Szabó 1998): symplectic surfaces (in sympl. 4-mflds) are genus minimizing in their homology class.

— also true when $\partial X = Y$ convex bdry, $\partial\Sigma = K$ transverse (Gadgil-Kulkarni 2020)

② $X = B^4$ $g_4(K) = \min\{g(\Sigma) \mid \Sigma \hookrightarrow B^4, \partial\Sigma = K\}$
slice genus

gauge theory ⇝ Milnor Conj. (Kronheimer-Mrowka 1993)
$g_4(T_{p,q}) = \frac{(p-1)(q-1)}{2}$

K is slice if $g_4(K) = 0$

X^4-closed

Def: $K \subset S^3$ is slice in X if $\exists\, \Delta \hookrightarrow X^\circ$ smooth disk, $\partial\Delta = K$.

K is H-slice in X if $\exists\, \Delta \hookrightarrow X^\circ$ smooth disk, $\partial\Delta = K$, $[\Delta] = 0 \in H_2(X)$.

Motivation: Smooth 4D Poincaré Conj.: $X^4 \simeq_{htpy} S^4 \Rightarrow X \cong_{diffeo.} S^4$

Strategy to disprove it: Find K that is slice ($\Leftrightarrow$ H-slice) in X but not slice (in S^4).

cf. Freedman-Gompf-Morrison-Walker (2009) tried this (using Rasmussen's s invariant)

Thm. (M-Marengon-Piccirillo) $\exists\, X, X'$ homeomorphic closed 4-mflds, $K \subset S^3$ H-slice in X but not in X'.

i.e. set of H-slice knots can detect exotic smooth structures.

Example: $X = 3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2}$, $X' = K3 \# \overline{\mathbb{CP}^2}$, K = RH trefoil.

Obs.: Previously, ∃ examples in 4-mflds with $\partial \neq S^3$ (e.g. Akbulut corks)

Background

$\mathcal{K}$ = {knots} $\mathcal{S}$ = {slice knots} (in S^4) $\mathcal{S}(X)$ = {slice knots in X} $\mathcal{S}_H(X)$ = {H-slice knots in X}

$$\mathcal{S} \subseteq \mathcal{S}_H(X) \subseteq \mathcal{S}(X) \subseteq \mathcal{K}$$

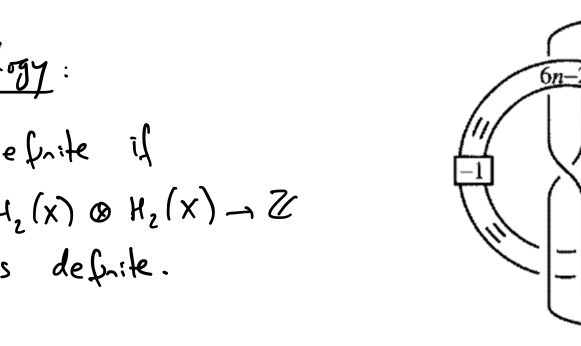
Examples:

$X = S^4, S^1\times S^3, T^4$: $\mathcal{S} = \mathcal{S}_H(X) = \mathcal{S}(X) \neq \mathcal{K}$

$X = S^2\times S^2$ or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$: $\mathcal{S} \neq \mathcal{S}_H(X) \neq \mathcal{S}(X) = \mathcal{K}$

$X = \mathbb{CP}^2$: $\mathcal{S} \neq \mathcal{S}_H(X) \neq \mathcal{S}(X) \neq \mathcal{K}$

RHT bounds Δ, $[\Delta]=0$; LHT: $\exists\Delta$, $[\Delta] = 2[\mathbb{CP}^1]$; $T_{2,-15}$ (Yasuhara)



$X = K3$ surface : $\mathcal{S} \neq \mathcal{S}_H(X) \neq \mathcal{S}(X) \overset{?}{=} \mathcal{K}$

e.g. Whitehead double of LHT; e.g. LHT or RHT: LHT bounds Δ, $[\Delta] \neq 0$, $[\Delta]^2 = 0$

Terminology:
X^4 – definite if $Q_X : H_2(X) \otimes H_2(X) \to \mathbb{Z}$ is definite.

$E(2) = K3$, $n=2$

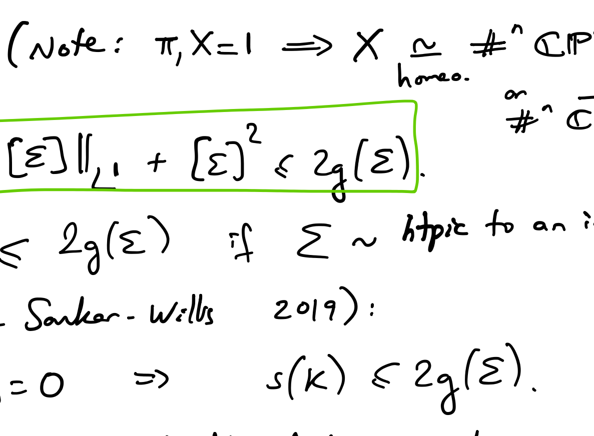


Figure 8.15. Elliptic surface $E(n)$.

Literature :

① Results for X- definite (Note: $\pi_1 X = 1 \Rightarrow X \simeq_{homeo.} \#^n \mathbb{CP}^2$ or $\#^n \overline{\mathbb{CP}^2}$)

Ozsváth-Szabó: $2\tau(K) + \|[\Sigma]\|_{L^1} + [\Sigma]^2 \leq 2g(\Sigma)$.

Kronheimer-Mrowka: $s^\#(K) \leq 2g(\Sigma)$ if $\Sigma \sim$ htpic to an inclusion in $\partial X^\circ = S^3$

Khovanov homology (M-Marengon-Sarkar-Willis 2019):
$X = \#^n \mathbb{CP}^2$, $[\Sigma] = 0 \Rightarrow s(K) \leq 2g(\Sigma)$.

② Topological results: OK for locally flat Σ in top. 4-mflds

Thm. 1 (Kirby?) $[\Sigma]$ - characteristic
$$\frac{\sigma(X) - [\Sigma]^2}{8} \equiv \text{Arf}(K) + \underbrace{\text{Arf}(X,\Sigma)}_{0 \text{ if } \Sigma = \text{disk}} + \underbrace{ks(X)}_{0 \text{ if } X\text{-smooth}} \mod 2$$

Robertello: K is H-slice in X (spin) $\Rightarrow \text{Arf}(K) = 0$.

e.g. RHT, LHT are not H-slice in K3.

Thm. 2 (Conway-Nagel) If $H_1(X) = 0$, $[\Sigma] = 0 \Rightarrow$
$$|\sigma_K(\omega) + \sigma(X)| \leq b_2(X) + 2g(\Sigma)$$
↖ Tristram-Levine signature, $\omega \in S^1$ (not a root of $p \in \mathbb{Z}[x, x^{-1}]$, $p(1) = 1$)

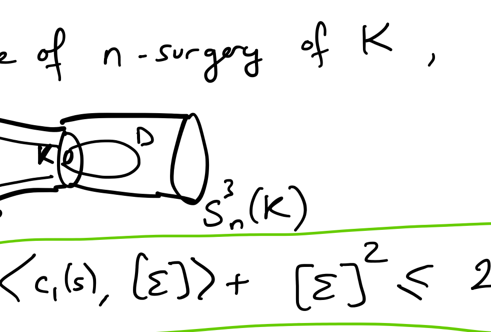
Thm. 3 (Viro-Gilmer): $H_1(X) = 0$, $m = p^k$ divides $[\Sigma]$ (p-prime)
$$\Rightarrow \left|\sigma_K(e^{2\pi r i/m}) + \sigma(X) - \frac{2r(m-r)}{m^2}[\Sigma]^2\right| \leq b_2(X) + 2g(\Sigma).$$

Cor. of Thm. 2 or 3: K is H-slice in K3 $\Rightarrow \sigma(K) = \sigma_K(-1) \in [-6, 38]$.

Focus: X = indefinite, $\Sigma \hookrightarrow X^\circ$ smooth

Adjunction inequality: X^4-closed, $b_2^+(X) > 1$, $\Sigma \hookrightarrow X$ closed, $g(\Sigma) > 0$
"of simple type"
$s \in \text{Spin}^c(X)$, $\Phi_{X,s} \neq 0 \Rightarrow$
$$\langle c_1(s), [\Sigma]\rangle + [\Sigma]^2 \leq 2g(\Sigma) - 2$$
↑ SW or OS 4-mfld. invt.

Relative case:



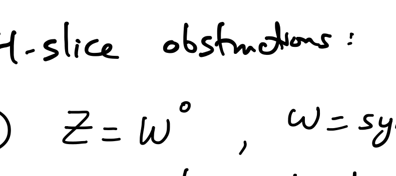
apply adjunction to $\Sigma \cup F \subset X^\circ \cup Z$

⇝ relative adjunction inequalities.

⓪ $Z = B^4$, F = min. genus surface for K

Thm. 0: $\langle c_1(s), [\Sigma]\rangle + [\Sigma]^2 \leq 2g(\Sigma) - 2 + 2g_4(K)$.

① Z = trace of n-surgery of K, $n \leq 0$



Thm. 1 (MMP): $\langle c_1(s), [\Sigma]\rangle + [\Sigma]^2 \leq 2g(\Sigma) - 2 + 2\nu^+(\overline{K})$
↑ Hom-Wu invariant from HFK, $0 \leq \nu^+ \leq g_4$; ↑ mirror

Ex.: K = LHT, X = K3, $s = 0$
Thm. 0 ⇒ If $g(\Sigma) = 0$, then $[\Sigma]^2 \leq 0$
Thm. 1 ⇒ If $g(\Sigma) = 1$, then $[\Sigma]^2 \leq 0$.

For H-slice obstructions:

② $Z = W^\circ$, W = sympl. mfld.

Thm. 2: X, X' = closed sympl. mflds. with $b_2^+ \equiv 3 \pmod 4$
$K \subset S^3$ s.t. $\overline{K}$ bounds a smooth disk $\Delta \subset X^\circ$
$[\Delta]^2 \geq 0$, $[\Delta] \neq 0$
Then K is not H-slice in X.

Sketch of proof: Suppose it were.
⇝ sphere $S \subset X \# X'$
$[S]^2 = 0$, $[S] \neq 0$
impossible in 4-mflds. with $SW \neq 0$ (Fintushel-Stern)
in our case: $X \# X'$ has $SW = 0$, but has $BF \neq 0$ (Bauer-Furuta)
stable homotopy refinement of SW
⇝ Fintushel-Stern still applies.

Cor.: RHT is H-slice in $X = \#3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2}$ but not in $X' = K3 \# \overline{\mathbb{CP}^2}$.

Pf.: H-slice in $\mathbb{CP}^2 \Rightarrow$ in X.
not H-slice in X' b/c LHT is H-slice in K3 (use Thm. 2)

Cor.: RHT is topologically but not smoothly H-slice in X'.

Another technique: Donald-Vafaee used Furuta's $\frac{10}{8}$ inequality to obstruct sliceness (in S^4).

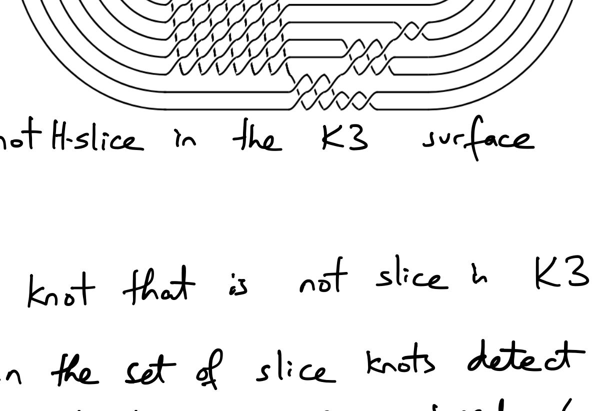
Generalization:

Thm. 3 (MMP): Suppose K = H-slice in X^4 – spin.
W = spin 2-handlebody $\partial W = S^3_0(K)$
If $b_2(X) + b_2(W) \neq 1, 3, 23$ then
$$b_2(X) + b_2(W) \geq \frac{10}{8}|\sigma(X) - \sigma(W)| + 5$$

$b_2 \geq \frac{10}{8}|\sigma| + 4$ if $b_2 \neq 0, 2, 22$, $X \sim S^4, S^2\times S^2, K3$

Sketch of proof: $V = X - (\text{slice disk})$ $\partial V = S^3_0(K)$
Apply $\frac{10}{8} + 4$ thm. (Hopkins-Lin-Shi-Xu) to $(-V) \cup W$.

Application: K_{DV} = topologically slice (in S^4), not slice (Donald-Vafaee)



Also, K_{DV} is not H-slice in the K3 surface (from Thm. 3)

Open questions:

① ∃ knot that is not slice in K3?

② Can the set of slice knots detect exotic smooth structures on some closed 4-mflds?

③ Is it the case that for every X^4 closed, $\exists\, K \subset S^3$ that is top. but not smoothly H-slice in X?