C*-Categories

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Interactions between C*-algebraic KK-theory
and homotopy theory

Definition of C*-categories

Originally due to Ghez–Lima–Roberts (1985).

Definition

Let **C** be a Banach $*$-category, i.e. morphism spaces $\mathrm{Hom}(x, y)$ are complex Banach spaces with norm $\| - \|_{x,y}$ s.t.

- $\|g \circ f\|_{x,y} \leq \|g\|_{y,z} \cdot \|f\|_{x,y}$ for all morphisms f and g, and
- the $*$-operation is by isometries.

C is a C*-category if additionally for all morphisms $f\colon x \to y$

- $\|f^* \circ f\|_{x,x} = \|f\|^2_{x,y}$ and
- $f^* \circ f$ is a positive element of the C*-algebra $\mathrm{Hom}(x, x)$.

Remark: The last point is indeed necessary (Schick).

Examples of C*-categories

1. Every C*-algebra is a C*-category with a single object.
2. $\mathbf{Hilb}(\mathbb{C})$ has as objects all (separable) Hilbert spaces and morphisms the bounded linear operators.
3. $\mathbf{Hilb}(\mathbb{C})$ contains the wide sub-C*-category $\mathbf{Hilb}_c(\mathbb{C})$, where we consider only the compact linear operators.
   (This is a non-unital C*-category.)
4. Generalizing the above, let A be a C*-algebra.
   $\mathbf{Hilb}(A)$ has as objects the Hilbert A-modules and as morphisms the bounded adjointable operators.
   Considering only the compact operators, we get $\mathbf{Hilb}_c(A)$.

The category of C*-categories

Definition

A functor $F \colon \mathbf{C} \to \mathbf{D}$ between C*-categories is a C*-functor if it is $\mathbb{C}$-linear and preserves the adjoint.

We will denote by **C*-Cat** the category whose objects are the C*-categories and morphisms the C*-functors.

The inclusions $\mathbf{Hilb}_c(A) \to \mathbf{Hilb}(A)$ are examples of C*-functors, i.e. morphisms in **C*-Cat**.

We also have a fully faithful functor **C*-Alg** $\to$ **C*-Cat**.

Why do we consider C*-categories?

We have the inclusion **C*-Alg** $\to$ **C*-Cat** and we would like to extend K-theory (or even KK-theory) along it.

This is usually done to fix problems (like functoriality) that arise from being forced to make choices in the constructions.

Example I: Roe algebras

Let X be a (proper) metric space. Choose a (separable) Hilbert space H and a suitable *-representation $\rho\colon C_0(X) \to \mathscr{B}(H)$.

We define a C*-algebra $C^*(X, H, \rho)$ as a certain sub-C*-algebra of $\mathscr{B}(H)$ — the Roe algebra — and K-theory of it should be the coarse K-homology of X.

For functoriality, let a suitable map $X \to Y$ between the metric spaces be given and fixed choices (H, ρ) for X and (H', ρ') for Y have been made. We choose a suitable isometry $V\colon H \to H'$ and conjugation by it gives a map between the Roe algebras of (X, H, ρ) and (Y, H', ρ').

So many choices ... fine on K-theory *groups* but not on *spectra*.

Example I: Roe algebras – solution

To solve the problem, we consider all possible choices of such pairs (H, ρ) at once.

We define a C*-category $C^*(X)$ whose objects are such pairs and whose morphisms are the analogous ones as in the Roe algebra – this is the Roe category of X.

For functoriality, if $f\colon X \to Y$ is any suitable map, we will get a C*-functor $C^*(X) \to C^*(Y)$ by mapping (H, ρ) to $(H, \rho \circ f^*)$ and on morphisms as the “identity”.

The coarse K-homology of X is the K-theory of this C*-category.

Example II: Functors on the orbit category

Davis–Lück: A functor $E\colon \mathbf{Or}(G) \to \mathbf{Sp}$ gives rise to an assembly map. Here the orbit category $\mathbf{Or}(G)$ has as objects the homogeneous G-spaces G/H and as morphisms G-maps.

For the Baum–Connes assembly map we would like to assign:

- To G/H the spectrum $K(C_r^*H)$.
- To a morphism $G/H \to G/K$, which is given by right multiplication $r_g\colon G/H \to G/K$, $g'H \mapsto g'gK$ provided $g \in G$ satisfies $g^{-1}HG \subset K$, the by $c_g\colon H \to K$, $h \mapsto g^{-1}hg$ induced morphism on reduced group C^*-algebras.[1]

  Problem: The choice of $g \in G$ is not unique since for $k \in K$ we have $r_g = r_{gk}$. But c_g and c_{gk} differ by conjugation by k which does not necessarily act trivially on $K(C_r^*K)$.

[1] Ignoring the functoriality issues with the reduced group C^*-algebra.

Example II: Functors on the orbit category – solution

We define a functor $\mathbf{Or}(G) \to$ **C*-Cat**:

- An object G/H is mapped to the C*-category with set of objects G/H and the morphism space for $g'H, g''H \in G/H$ is „generated" by elements $l \in G$ with $lg'H = g''H$.
- A morphism $r_g \colon G/H \to G/K$, $g'H \mapsto g'gK$ is mapped to the C*-functor which is the morphism itself on objects and the „identity" on the generators l.

Now we can apply our extension of K-theory to C*-categories and have our sought functor on the orbit category.

Extending *K*- and *KK*-theory to C*-categories

Let us now extend *K*-theory (and *KK*-theory) along the inclusion **C*-Alg** $\rightarrow$ **C*-Cat**. The idea explained now is due to M. Joachim.

This inclusion has a left adjoint A^f: **C*-Cat** $\rightarrow$ **C*-Alg** and the extension is then just the composition

$$\textbf{C*-Cat} \xrightarrow{A^f} \textbf{C*-Alg} \rightarrow \textbf{KK}$$

(resp. using the *K*-theory functor **C*-Alg** $\rightarrow$ **Sp**).

The nature of this functor A^f will be explained later.

Introduction to unitary equivalences

In the above discussed two examples, the different choices that we can make are related by conjugation by unitaries (or at least isometries).

Therefore we don't have any problems classically: Conjugations by unitaries act trivially on the K-theory *groups*. But they usually don't act trivially on the *spectra*.

We will see now that the axioms for KK-theory alone suffice to prove that it sends unitarily equivalent C*-functors to equivalent morphisms.

Definition of unitary equivalences

Definition

Let $\mathbf{C}, \mathbf{D}$ be C*-categories and let $F, G\colon \mathbf{C} \to \mathbf{D}$ be C*-functors.

They are unitarily equivalent if there is a natural isomorphism $\eta\colon F \to G$ acting unitarily, i.e. for a morphism $\varphi \in \mathrm{Hom}_{\mathbf{C}}(c, c')$ the vertical maps in the following diagram are unitaries:

$$\begin{array}{ccc} F(c) & \xrightarrow{F(\varphi)} & F(c') \\ {\scriptstyle \eta_c}\downarrow & & \downarrow{\scriptstyle \eta_{c'}} \\ G(c) & \xrightarrow[G(\varphi)]{} & G(c') \end{array}$$

Using unitary equivalences, we can therefore turn **C*-Cat** naturally into a (2,1)-category.

Respecting unitary equivalences follows from axioms

Proposition

*Let $E\colon$ **C*-Alg** $\to$ **S** be a matrix-stable and homotopy invariant functor, where **S** is an ∞-category.*

*Then $E \circ A^f\colon$ **C*-Cat** $\to$ **S** sends unitarily equivalent C*-functors to equivalent morphisms that coincide in the homotopy category.*

The functor A^f made concrete

Definition

Let **C** be a C*-category.

We consider the *-algebra freely generated by all the morphisms of **C**, and then form the quotient by an equivalence relation which reflects the sum, composition and the *-operation present in **C**. Completing this in the maximal C*-norm defines $A^f(\mathbf{C})$.

We will denote the generators of $A^f(\mathbf{C})$ by (φ), where φ is any morphism in **C**.

Proof of the proposition

Let $F, G\colon \mathbf{C} \to \mathbf{D}$ be unitarily equivalent C*-functors. We want to show that $(E \circ A^f)(F)$ is equivalent to $(E \circ A^f)(G)$.

Let η be the corresponding unitary equivalence with components $\eta_c\colon F(c) \to G(c)$ for objects c in $\mathbf{C}$, and let (φ) be a generator of $A^f(\mathbf{C})$ for $\varphi\colon c \to c'$. The following is a homotopy in $\mathbb{M}_2 \otimes A^f(\mathbf{D})$ between $\mathrm{diag}(0, (F(\varphi)))$ and $\mathrm{diag}((G(\varphi)), 0)$:

$$\begin{pmatrix} (\eta_{c'}) & 0 \\ 0 & 1 \end{pmatrix} \cdot u_t \cdot \begin{pmatrix} (F(\varphi)) & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} (\eta_c^{-1}) & 0 \\ 0 & 1 \end{pmatrix} \cdot u_t^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & (\eta_c) \end{pmatrix},$$

where $u_t := \begin{pmatrix} \cos(\pi/2 \cdot t) & -\sin(\pi/2 \cdot t) \\ \sin(\pi/2 \cdot t) & \cos(\pi/2 \cdot t) \end{pmatrix}$.

Now apply these homotopies simultaneously to each generator to get a homotopy from $\mathrm{diag}(A^f(G), 0)$ to $\mathrm{diag}(0, A^f(F))$. $\square$

Thanks for your attention!