Morse-Bott theory on analytic spaces and applications to the topology of smooth 4-manifolds

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Outline

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Collaborators and references

This talk is based on joint work with **Tom Leness** in our paper [4]:

- *Introduction to virtual Morse–Bott theory on analytic spaces, moduli spaces of* SO(3) *monopoles, and applications to four-manifolds* (with T. Leness), arXiv:2010.15789.

Work in progress (a specialization of our program to Kähler surfaces, not discussed today) is joint with Leness and **Richard Wentworth**.

The *first part* of our talk will focus on the more expository Sections 1 through 5.

The *second part* of our talk will focus on the more technical Section 6.

Introduction

Introduction I

We shall describe a new approach to Morse–Bott theory, called *virtual Morse–Bott theory*, that applies to singular (real or complex) analytic spaces that arise in gauge theory, including moduli spaces of

- $SO(3)$ monopoles over closed smooth four-manifolds of Seiberg–Witten simple type,
- Stable holomorphic pairs of bundles and sections over closed complex Kähler surfaces,
- Higgs pairs over closed Riemann surfaces.

The moduli spaces over Kähler surfaces or Riemann surfaces are *complex analytic spaces*, with Kähler metrics and Hamiltonian circle actions.

Introduction II

For a smooth four-manifold that is *almost Hermitian* (which includes four-manifolds of *Seiberg–Witten simple type*), where

- the *almost complex structure* is not necessarily integrable and
- the *fundamental two-form* defined by the almost complex structure and Riemannian metric is not necessarily closed,

one can still show that the moduli space of SO(3) monopoles is a *real analytic space* and (after some work) that it is *almost Hermitian* [5].

These moduli spaces carry a circle action compatible with the almost complex structure and Riemannian metric and a Hamiltonian function to which virtual Morse–Bott theory applies.

We shall outline how virtual Morse–Bott theory may help prove the

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Introduction III

Conjecture 1.1 (Bogomolov–Miyaoka–Yau (BMY) inequality for four-manifolds with non-zero Seiberg–Witten invariants)

If X is a closed, oriented, smooth four-manifold with $b_1(X) = 0$, odd $b^+(X) \geq 3$, and Seiberg–Witten simple type with a non-zero Seiberg–Witten invariant, then

$$c_1(X)^2 \leq 9\chi_h(X). \tag{1}$$

Yau [31] proved (1) for a compact Kähler surface X with ample canonical bundle using his existence of a Kähler–Einstein metric whose Ricci curvature is a negative constant [32] and a Chern–Weil inequality [30].

Inequality (1) was proved by Miyaoka [21] using algebraic geometry.

Introduction IV

If X obeys the hypotheses of Conjecture 1.1, then it has an almost complex structure J [17] and in the inequality (1), which is equivalent to

$$c_1(X)^2 \leq 3c_2(X),$$

the Chern classes are those of the complex vector bundle (TX, J).

Taubes [27, 28] showed that *symplectic four-manifolds* have Seiberg–Witten simple type and so they satisfy the hypotheses of Conjecture 1.1.

Frankel's Theorem for the Hamiltonian function for a circle action on a smooth almost Hermitian manifold

Frankel's Theorem for almost Hermitian manifolds I

The version of Frankel's Theorem [9, Section 3] that we prove and apply in [4] is a little more general because we allow for circle actions on closed, smooth manifolds (M, g, J) that are only assumed to be *almost Hermitian*, rather than (almost) Kähler:

- the almost complex structure J need not be integrable and
- the fundamental two-form $\omega = g(\cdot, J\cdot)$ defined by the compatible pair (g, J) is non-degenerate but not required to be closed.

Frankel assumed in [9, Section 3] that ω was closed (though he allowed J to be non-integrable [9, p. 1].

Recall that $J \in C^\infty(\mathrm{End}_{\mathbb{R}}(TM))$ is an *almost complex structure* on M if

$$J^2 = -\mathrm{id}_{TM}$$

Frankel's Theorem for almost Hermitian manifolds II

and J is *orthogonal with respect to* or *compatible with* a Riemannian metric g on M if

$$g(JX, JY) = g(X, Y)$$

for all vector fields $X, Y \in C^\infty(TM)$.

Frankel's Theorem for almost Hermitian manifolds III

Theorem 1 (Frankel's Theorem for circle actions on almost Hermitian manifolds)

(Compare Frankel [9, Section 3].) Let (M, g, J) be a finite-dimensional, smooth, **almost Hermitian manifold** *with fundamental two-form $\omega = g(\cdot, J\cdot)$. Assume that M has a smooth* **circle action** *$\rho : S^1 \times M \to M$ and let $\rho_* : S^1 \times TM \to TM$ denote the induced circle action on the tangent bundle TM given by $\rho_*(e^{i\theta})v = D_2\rho(e^{i\theta}, p)v$, for all $v \in T_pM$ and $e^{i\theta} \in S^1$. Assume that the circle action is orthogonal with respect to g and compatible with J in the sense that*

$$g\left(\rho_*(e^{i\theta})v, \rho_*(e^{i\theta})w\right) = g(v, w) \quad \text{and} \quad J\rho_*(e^{i\theta})v = \rho_*(e^{i\theta})Jv,$$
$$\text{for all } p \in M, v, w \in T_pM, \text{ and } e^{i\theta} \in S^1.$$

Assume further that the circle action is **Hamiltonian** *in the sense that there exists a function $f \in C^\infty(M, \mathbb{R})$ such that $df = \iota_X\omega$, where $X \in C^\infty(TM)$ is the vector field generated by the circle action, so $X_p = D_1\rho(1, p)$ for all $p \in M$ and*

$$\iota_X\omega(Y) = \omega(X, Y) = g(X, JY), \quad \text{for all } Y \in C^\infty(TM).$$

Frankel's Theorem for almost Hermitian manifolds IV

Theorem 1 (Frankel's Theorem for circle actions on almost Hermitian manifolds)

If $p \in M$ is a **critical point** *of the* **Hamiltonian function** *f (equivalently, a* **fixed point** *of the circle action), then*

- *the* **eigenvalues** *of the Hessian $\operatorname{Hess}_g f \in \operatorname{End}(T_pM)$ of f are given by the* **weights** *of the circle action on T_pM,*
- *f is* **Morse–Bott** *at p in the sense that in a small enough open neighborhood of p, the critical set $\operatorname{Crit} f := \{q \in M : df(q) = 0\}$ is a smooth submanifold with tangent space $T_p \operatorname{Crit} f = \operatorname{Ker} \operatorname{Hess}_g f(p)$, and*
- *each connected component of $\operatorname{Crit} f$ has even dimension and even codimension in M.*

We prove Theorem 1 and further extensions in [4].

Frankel's Theorem for almost Hermitian manifolds V

The *gradient vector field* $\operatorname{grad}_g f$ on M for a smooth function $f : M \to \mathbb{R}$ is defined by

$$g(\operatorname{grad}_g f, Y) := df(Y), \quad \text{for all } Y \in C^\infty(TM).$$

If ∇^g is the Levi–Civita connection on TM, the *Hessian of f* is

$$\operatorname{Hess}_g f := \nabla^g \operatorname{grad}_g f \in C^\infty(\operatorname{End}(TM)).$$

Theorem 1 implies that the following are equal:

- Subspace $T_p^- M \subset T_p M$ on which $\operatorname{Hess}_g f(p) \in \operatorname{End}(T_p M)$ is *negative definite*,
- Subspace of $T_p M$ on which S^1 acts with *negative weight*.

Hence, the *(classical) Morse–Bott index* of f at a critical point p, by definition $\dim T_p^- M$, equals the dimension of the subspace of $T_p M$ on which the circle acts with *negative weight*.

Virtual Morse–Bott index for the Hamiltonian function of a circle action on an analytic space

Virtual Morse–Bott index for analytic spaces I

Hitchin's results [12, Proposition 7.1 and Theorem 7.6] for the

1. Critical submanifolds and Morse indices of a Hamiltonian function
2. Topology of moduli space of Higgs bundles over a Riemann surface

show that Frankel's Theorem 1 is remarkably powerful.

One goal of our article [4] is to show that Frankel's Theorem 1 has useful generalizations to analytic spaces that are *singular*.

Analytic spaces are locally isomorphic to analytic varieties in $\mathbb{K}^n$ — zero sets of finitely many $\mathbb{K}$-analytic functions for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

We first state the simpler of two results in virtual Morse–Bott theory.

Virtual Morse–Bott index for analytic spaces II

Theorem 2 (Virtual Morse–Bott index of a critical point of a real analytic function on a real analytic space)

Let X be a finite-dimensional real analytic manifold, $M \subset X$ be a real analytic subspace, $p \in M$ be a point, and $F : \mathscr{U} \to \mathbb{R}^n$ be an analytic local defining function for M on an open neighborhood $\mathscr{U}$ of p in the sense that $M \cap \mathscr{U} = F^{-1}(0) \cap \mathscr{U}$. Let $T_pM = \operatorname{Ker} dF(p)$ denote the Zariski tangent space to M at p. Let $f : X \to \mathbb{R}$ be an analytic function and assume that p is a **Morse–Bott critical point** *of the restriction $f : M \to \mathbb{R}$ in the sense that*

- $\mathscr{C} = \{q \in M \cap \mathscr{U} : \operatorname{Ker} df(q) = T_qM\}$ *is a real analytic submanifold of X, and*
- $T_p\mathscr{C} = \operatorname{Ker} \operatorname{Hess} f(p)$.

Let $\operatorname{Ker}^{\pm} dF(p) = T_p^{\pm}M \subset T_pM$ denote the maximal positive and negative real subspaces for $\operatorname{Hess} f(p) \in \operatorname{End}(T_pM)$. If the **virtual Morse–Bott index**

$$\lambda_p^{-}(f) := \dim \operatorname{Ker}^{-} dF(p) - \dim \operatorname{Coker} dF(p), \tag{2}$$

is **positive**, *then p is* **not a local minimum** *for $f : M \to \mathbb{R}$.*

Virtual Morse–Bott index for analytic spaces III

The following gives a generalization of Theorem 1, specialization of Theorem 2, and refinement of the definition of virtual Morse–Bott index.

Theorem 3 (Virtual Morse–Bott index of a critical point of a Hamiltonian function for a circle action on a complex analytic space)

Let X be a complex, finite-dimensional, Kähler manifold with circle action that is compatible with the complex structure and induced Riemannian metric. Assume that the circle action is Hamiltonian with analytic Hamiltonian function $f : X \to \mathbb{R}$ such that $df = \iota_\xi \omega$, where ω is the Kähler form on X and ξ is the vector field on X generated by the circle action. Let $M \subset X$ be a closed, complex analytic subspace, $p \in M$ be a point, and $F : \mathscr{U} \to \mathbb{C}^n$ be an analytic, circle equivariant, local defining function for M on an open neighborhood $\mathscr{U} \subset X$ of p in the sense that $M \cap \mathscr{U} = F^{-1}(0) \cap \mathscr{U}$. Let

- $\mathbf{H}_p^2 \subset \mathbb{C}^n$ *denote the orthogonal complement of* $\operatorname{Ran} dF(p) \subset \mathbb{C}^n$,
- $\mathbf{H}_p^1 = \operatorname{Ker} dF(p) \subset T_pX$ *denote the Zariski tangent space to M at p, and*
- $M^{\mathrm{vir}} \subset X$ *denote the complex, Kähler submanifold given by $F^{-1}(\mathbf{H}_p^2) \cap \mathscr{U}$ and observe that $T_pM^{\mathrm{vir}} = \mathbf{H}_p^1$.*

Virtual Morse–Bott index for analytic spaces IV

Theorem 3 (Virtual Morse–Bott index of a critical point of a Hamiltonian function for a circle action on a complex analytic space)

If p is a critical point of $f : M \to \mathbb{R}$ in the sense that $\operatorname{Ker} df(p) = \mathbf{H}_p^1$, then p is a fixed point of the induced circle action on M^{vir}.

- *Let $S \subset M^{\mathrm{vir}}$ be the connected component containing p of the complex analytic submanifold of M^{vir} given by the set of fixed points of the circle action on M^{vir} and assume that $S \subset M$.*
- *Let $\mathbf{H}_p^{1,-} \subset \mathbf{H}_p^1$ and $\mathbf{H}_p^{2,-} \subset \mathbf{H}_p^2$ denote the subspaces on which the circle acts with* **negative weight**.

If the **virtual Morse–Bott index**

$$\lambda_p^-(f) := \dim_{\mathbb{R}} \mathbf{H}_p^{1,-} - \dim_{\mathbb{R}} \mathbf{H}_p^{2,-} \tag{3}$$

is **positive**, *then p is* **not a local minimum** *for $f : M \to \mathbb{R}$.*

Virtual Morse–Bott index for analytic spaces V

To prove prove Theorems 2 and 3, we use the

- Embedded Resolution of Singularities Theorem for (real or complex) analytic spaces (see Hironaka [11]), and
- Generic perturbation and transversality arguments.

Extensions and generalizations of Theorem 3

As we show in [4, 5], Theorem 3 generalizes to the case where X is a **real analytic, almost Hermitian manifold**.

That statement and its proof are provided in [4, 5].

Theorem 3 suffices for applications to the moduli spaces considered in [4] and this talk and whose top strata of smooth points are known to be complex Kähler manifolds.

Moduli spaces of anti-self-dual connections, Seiberg–Witten monopoles, and SO(3) monopoles

ASD connections, Seiberg–Witten & SO(3) monopoles I

Four-manifold topology

For a closed topological four-manifold X, we define

$$c_1(X)^2 := 2e(X) + 3\sigma(X) \quad \text{and} \quad \chi_h(X) := \frac{1}{4}(e(X) + \sigma(X)),$$

where $e(X) = 2 - 2b_1(X) + b_2(X)$ and $\sigma(X) = b^+(X) - b^-(X)$ are the *Euler characteristic* and *signature* of X, respectively.

We call X **standard** if it is closed, connected, oriented, and smooth with odd $b^+(X) \geq 3$ and $b_1(X) = 0$.

If Q_X is the intersection form on $H_2(X;\mathbb{Z})$, then $b^{\pm}(X)$ are the dimensions of the maximal positive and negative subspaces of Q_X on $H_2(X;\mathbb{R})$.

ASD connections, Seiberg–Witten & SO(3) monopoles II

Seiberg–Witten monopoles

For a standard four-manifold X, its *Seiberg–Witten invariants* define

$$\mathrm{SW}_X : \mathrm{Spin}^c(X) \ni \mathfrak{s} \mapsto \mathrm{SW}_X(\mathfrak{s}) \in \mathbb{Z}.$$

A *spin^c structure* $\mathfrak{s} = (\rho, W)$ is a pair of rank-2 Hermitian vector bundles $W^{\pm}$ over X with $W = W^+ \oplus W^-$ and a *Clifford multiplication map*, $\rho : T^*X \to \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$.

The *Seiberg–Witten moduli space* $M_{\mathfrak{s}}$ is the set of gauge-equivalence classes of solutions to the *Seiberg–Witten* U(1)*-monopole equations* and is an orientable, compact, finite-dimensional, smooth manifold [22, 23, 26].

A *Seiberg–Witten invariant* $\mathrm{SW}_X(\mathfrak{s})$ is defined by counting signed points in $M_{\mathfrak{s}}$ when $\dim M_{\mathfrak{s}} = 0$ or pairing a natural cohomology class with $[M_{\mathfrak{s}}]$ when $\dim M_{\mathfrak{s}} > 0$.

ASD connections, Seiberg–Witten & SO(3) monopoles III

The set of *Seiberg–Witten basic classes* is finite:

$$B(X) := \left\{c_1(W^+) : \mathrm{SW}_X(\mathfrak{s}) \neq 0\right\} \subset H^2(X;\mathbb{Z}).$$

X has *Seiberg–Witten simple type* if $\dim M_{\mathfrak{s}} = 0$ for all $K \in B(X)$.

Known standard 4-manifolds have simple type (see [18, Conjecture 1.6.2]).

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ASD connections, Seiberg–Witten & SO(3) monopoles IV

Anti-self-dual connections

For $w \in H^2(X;\mathbb{Z})$ and $4\kappa \in \mathbb{Z}$, let E be a rank-2 Hermitian bundle over X with $c_1(E) = w$ and Pontrjagin number $p_1(\mathfrak{su}(E)) = -4\kappa$, where $\mathfrak{su}(E) \subset \mathfrak{gl}(E)$ is the SO(3) subbundle of trace-zero, skew-Hermitian endomorphisms of E.

The *moduli space of projectively anti-self-dual (ASD) connections* on E is

$$M_\kappa^w(X,g) := \{A \in \mathscr{A}_E : (F_A^+)_0 = 0\}/\mathscr{G}_E,$$

where $\mathscr{A}_E$ is the Banach affine space of fixed-determinant, unitary connections A on E, and F_A^+ is the self-dual component defined by a metric g on X of the curvature F_A of A, and $(F_A^+)_0$ is the trace-free component of F_A^+, and $\mathscr{G}_E$ is the Banach Lie group of determinant-one, unitary automorphisms of E.

The space $M_\kappa^w(X,g)$ is an oriented smooth manifold [1, 10] for generic g.

ASD connections, Seiberg–Witten & SO(3) monopoles V

SO(3) monopoles

Let $\mathfrak{t} = (\rho, W, E)$ be a *spin^u structure* over X and $\tilde{\mathscr{C}}_{\mathfrak{t}}$ be the space of pairs (A, Φ) of fixed-determinant, unitary connections A on a Hermitian rank-two vector bundle E and sections Φ of $W^+ \otimes E$.

We call $(A, \Phi) \in \tilde{\mathscr{C}}_{\mathfrak{t}}$ an SO(3) *monopole* if

$$\mathfrak{S}(A, \Phi) := \begin{pmatrix} (F_A^+)_0 - \rho^{-1}(\Phi \otimes \Phi^*)_{00} \\ D_A \Phi \end{pmatrix} = 0, \qquad (4)$$

where the section $(\Phi \otimes \Phi^*)_{00}$ of $\mathfrak{su}(W^+) \otimes \mathfrak{su}(E)$ is the trace-free component of $\Phi \otimes \Phi^*$ of $\mathfrak{u}(W^+) \otimes \mathfrak{u}(E)$ and D_A is the Dirac operator and $\rho : \wedge^+(T^*X) \to \mathfrak{su}(W^+)$ is an isomorphism of SO(3) bundles.

ASD connections, Seiberg–Witten & SO(3) monopoles VI

The *moduli space of* SO(3) *monopoles* is

$$\mathscr{M}_{\mathfrak{t}} := \left\{ (A, \Phi) \in \tilde{\mathscr{C}}_{\mathfrak{t}} : \text{Equation (4) holds} \right\}$$

and it has a decomposition as a disjoint union of subsets

$$\mathscr{M}_{\mathfrak{t}} = \mathscr{M}_{\mathfrak{t}}^{*,0} \sqcup \mathscr{M}_{\mathfrak{t}}^{\{\Phi \equiv 0\}} \sqcup \mathscr{M}_{\mathfrak{t}}^{\{A \text{ reducible}\}},$$

where $\mathscr{M}_{\mathfrak{t}}^{*,0} \subset \mathscr{M}_{\mathfrak{t}}$ is the subspace of *irreducible, non-zero-section pairs* and a *finite-dimensional smooth manifold* for generic geometric perturbations [6, 3, 29].

ASD connections, Seiberg–Witten & SO(3) monopoles VII

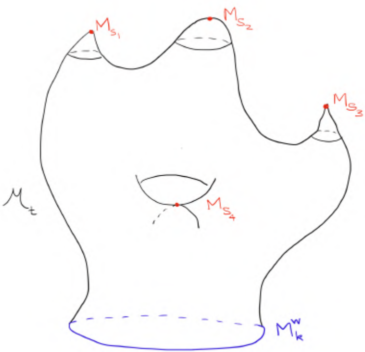

Figure 4.1: SO(3) monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with Seiberg–Witten moduli subspaces $\cup_i M_{\mathfrak{s}_i} \cong \mathscr{M}_{\mathfrak{t}}^{\{A \text{ reducible}\}}$ and moduli subspace $M_\kappa^w(X,g) \cong \mathscr{M}_{\mathfrak{t}}^{\{\Phi\equiv 0\}}$ of anti-self-dual connections

ASD connections, Seiberg–Witten & SO(3) monopoles VIII

The circle action on sections Φ induces a circle action on $\mathscr{M}_{\mathfrak{t}}$ with two types of fixed points, represented by pairs (A, Φ) such that

- $\Phi \equiv 0$ or
- A is a reducible connection.

For points $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$, there are bijections between

- the subset of $\mathscr{M}_{\mathfrak{t}}$ where $\Phi \equiv 0$ and the moduli space $M_{\kappa}^{w}(X, g)$ of ideal ASD connections and
- subsets of $\mathscr{M}_{\mathfrak{t}}$ where A is reducible with respect to splittings, $E = L_1 \oplus L_2$, and Seiberg–Witten moduli spaces $M_{\mathfrak{s}}$ defined by $\mathfrak{s} = (\rho, W \otimes L_1)$.

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ASD connections, Seiberg–Witten & SO(3) monopoles IX

The space $\mathscr{M}_{\mathfrak{t}}$ is noncompact due to *energy bubbling*, but admits an *Uhlenbeck compactification* $\bar{\mathscr{M}}_{\mathfrak{t}}$.

(The subspace $\mathscr{M}_{\mathfrak{t}}^{\{\Phi\equiv 0\}} = M_\kappa^w(X,g)$ is noncompact as well and also admits an Uhlenbeck compactification, $\bar{M}_\kappa^w(X,g)$.)

We discuss the complications due to noncompactness in [4, 5].

Virtual Morse–Bott theory, existence of anti-self-dual connections, and the Bogomolov–Miyaoka–Yau inequality

Virtual Morse–Bott theory, ASD connections, and BMY I

The *expected dimension* of the moduli space $M_\kappa^w(X,g)$ of g-anti-self-dual connections on $\mathfrak{su}(E)$ is given by [1]

$$\dim M_\kappa^w(X,g) = -2p_1(\mathfrak{su}(E)) - 6\chi_h(X). \tag{5}$$

When g is *generic* in the sense of [1, 10], then $M_\kappa^w(X,g)$ is an open, smooth manifold if non-empty.

We now suppose the topology of E is constrained by a *basic lower bound*,

$$p_1(\mathfrak{su}(E)) \geq c_1(X)^2 - 12\chi_h(X), \tag{6}$$

and ask whether existence of a spin^c structure $\mathfrak{s}$ over X with *non-zero Seiberg–Witten invariant* $\mathrm{SW}_X(\mathfrak{s})$ implies that $M_\kappa^w(X,g)$ is non-empty?

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Virtual Morse–Bott theory, ASD connections, and BMY II

For now, suppose that $\mathfrak{su}(E)$ does admit an *ASD connection* when inequality (6) holds and that the metric g on X is generic.

Then the moduli space $M_\kappa^w(X,g)$ of ASD connections on $\mathfrak{su}(E)$ with $w = c_1(E)$ and $\kappa = -\frac{1}{4}p_1(\mathfrak{su}(E))$ is a non-empty, smooth manifold so

$$\dim M_\kappa^w(X,g) \geq 0.$$

Therefore,

$$\begin{aligned}0 &\leq \frac{1}{2}\dim M_\kappa^w(X,g) \\ &= -p_1(\mathfrak{su}(E)) - 3\chi_h(X) \quad \text{(by (5))} \\ &\leq -\left(c_1(X)^2 - 12\chi_h(X)\right) - 3\chi_h(X) \quad \text{(by (6))} \\ &= -c_1(X)^2 + 9\chi_h(X),\end{aligned}$$

and this yields the Bogomolov–Miyaoka–Yau inequality (1).

Virtual Morse–Bott theory, ASD connections, and BMY III

Our strategy uses *virtual Morse-Bott theory* to prove existence of ASD connections on $\mathfrak{su}(E)$ for generic Riemannian metrics g on X when the basic lower bound (6) holds and $\mathrm{SW}_X(\mathfrak{s}) \neq 0$ for some spin^c structure $\mathfrak{s}$.

This approach relies on our link pairing formulae [8] to prove that $\mathscr{M}_{\mathfrak{t}}^{*,0}$ is non-empty.

Virtual Morse–Bott theory on the moduli space of SO(3) monopoles over a closed, complex Kähler surface

Outline of the strategy I

Virtual Morse–Bott theory on the *singular* moduli space $\mathscr{M}_\mathfrak{t}$ of SO(3) monopoles is a partial extension and adaptation of Hitchin's Morse–Bott theory on the *smooth* moduli space of Higgs pairs [12].

We use the following analogue of Hitchin's Morse function [12]:

$$f : \mathscr{M}_\mathfrak{t} \ni [A, \Phi] \mapsto f[A, \Phi] = \frac{1}{2}\|\Phi\|^2_{L^2(X)} \in \mathbb{R}. \tag{7}$$

This function is smooth, but not necessarily Morse–Bott on $\mathscr{M}_\mathfrak{t}$.

We say that a point $p = [A, \Phi] \in \mathscr{M}_\mathfrak{t}$ is a *critical point* of f if $df(p) \equiv 0$ on the Zariski tangent space $T_p\mathscr{M}_\mathfrak{t}$.

Outline of the strategy II

Outline of strategy

1. Prove existence of spin^u structure $\mathfrak{t} = (\rho, W, E)$ with
   - $p_1(\mathfrak{su}(E))$ obeying the basic lower bound (6) and having a
   - non-empty SO(3) monopole moduli subspace $\mathscr{M}_{\mathfrak{t}}^{*,0}$ of irreducible, non-zero-section pairs.
2. Prove that all critical points of Hitchin's function on $\mathscr{M}_{\mathfrak{t}}$ are
   - points in the ASD moduli subspace $M_{\kappa}^{w}(X, g) \subset \mathscr{M}_{\mathfrak{t}}$, or
   - points in moduli subspaces $M_{\mathfrak{s}} \subset \mathscr{M}_{\mathfrak{t}}$ of Seiberg–Witten monopoles with *positive virtual Morse–Bott index*.

Outline of the strategy III

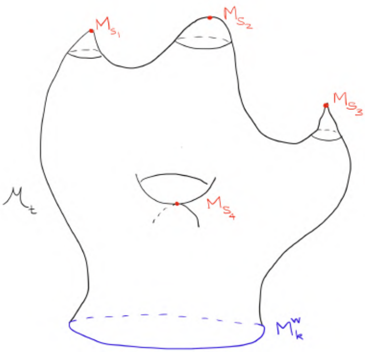

Figure 6.2: SO(3) monopole moduli space $\mathscr{M}_{\mathfrak{t}}$ with Seiberg–Witten moduli subspaces $M_{\mathfrak{s}_i}$ and moduli subspace $M_{\kappa}^{w}(X, g)$ of anti-self-dual connections

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This Concludes Part 1 of Our Talk

Complications in extending Hitchin's paradigm I

Part 2

In contrast to the moduli space of Higgs pairs analyzed by Hitchin [12], we must address two fundamental new difficulties in our application:

1. The strata $M_{\mathfrak{s}} \subset \mathscr{M}_{\mathfrak{t}}$ of moduli subspaces of Seiberg–Witten monopoles are smooth submanifolds but **not necessarily smoothly embedded as submanifolds** of $\mathscr{M}_{\mathfrak{t}}$; and
2. The moduli space $\mathscr{M}_{\mathfrak{t}}$ of SO(3) monopoles is **non-compact due to energy bubbling**.

Complications in extending Hitchin's paradigm II

Hitchin [12] assumes that the degree of the Hermitian vector bundle E is odd and that its rank is two.

More generally, if the degree and rank of the Hermitian vector bundle E in the equations (5.1) for a Higgs pair are *coprime*, then

1. Hitchin's moduli space of Higgs pairs is a finite-dimensional, analytic manifold,
2. The strata of reducible Higgs pairs are smoothly embedded submanifolds, and
3. f is Morse–Bott along the strata of reducible pairs with positive (classical) Morse–Bott index.

Feasibility of the SO(3)-monopole cobordism method I

The first step in our program is to construct a spin^u structure $\mathfrak{t} = (\rho, W, E)$ with $p_1(\mathfrak{su}(E))$ satisfying (6) and for which $\mathscr{M}_{\mathfrak{t}}^{*,0}$ is non-empty so that the gradient flow of (1.4) will have a starting point.

To obtain greater control over the characteristic classes of the spin^u structure, we work on the blow-up $\widetilde{X} := X\#\overline{\mathbb{CP}}^2$ of X.

Because $c_1(\widetilde{X})^2 = c_1(X)^2 - 1$, we replace the basic lower bound (6) with the equivalent bound $p_1 \geq c_1(\widetilde{X}) + 1 - 12\chi_h(\widetilde{X})$. We then prove

Feasibility of the SO(3)-monopole cobordism method II

Theorem 4 (Feasibility of spin^u structures; positivity of virtual Morse–Bott indices)

Let X be a standard four-manifold of Seiberg–Witten simple type, let $\widetilde{X} = X \# \overline{\mathbb{CP}}^2$ denote the blow-up of X, and let $\tilde{\mathfrak{s}}$ be a spin^c structure over X with non-zero Seiberg–Witten invariant $SW_{\widetilde{X}}(\tilde{\mathfrak{s}})$. Then there exists a spin^u structure $\tilde{\mathfrak{t}}$ over $\widetilde{X}$ such the following hold:

1. $M_{\tilde{\mathfrak{s}}} \subset \mathscr{M}_{\tilde{\mathfrak{t}}}$;
2. *The moduli space* $\mathscr{M}_{\tilde{\mathfrak{t}}}^{*,0}$ *is non-empty; and*
3. $p_1(\tilde{\mathfrak{t}}) \geq c_1(\widetilde{X})^2 + 1 - 12\chi_h(\widetilde{X})$.

Moreover, for all non-empty Seiberg–Witten moduli subspaces $M_{\mathfrak{s}'} \subset \mathscr{M}_{\tilde{\mathfrak{t}}}$, the virtual Morse–Bott index of the Hitchin function f in (7) along $M_{\mathfrak{s}'}$ is ***positive****.*

While we prove Theorem 6 in [4] for closed complex Kähler surfaces, the calculations extend to standard four-manifolds.

Critical points in space of SO(3) monopoles I

The next step in our program is to use Frankel's Theorem (see Theorem 1) to identify the critical points of Hitchin's function on $\mathscr{M}_{\mathfrak{t}}$.

In this talk, we shall restrict our discussion to SO(3) monopoles over closed complex Kähler surfaces.

Suppose that (X, g, J) is a complex Kähler surface, with Kähler form $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$.

A version of the **Hitchin–Kobayashi bijection** then identifies the moduli space $\mathscr{M}_{\mathfrak{t}}$ of SO(3) monopoles with the moduli space of *stable holomorphic pairs* (see Dowker [2], Lübke and Teleman [19, 20], and Okonek and Teleman [24, 25]).

Thus $\mathscr{M}_{\mathfrak{t}}$ is a *complex analytic space* and the open subspace $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}} \subset \mathscr{M}_{\mathfrak{t}}$ of smooth points is a *complex manifold*, generalizing results by Itoh

Critical points in space of SO(3) monopoles II

[13, 14] for the complex structure on the moduli space $M_\kappa^w(X,g)$ of anti-self-dual connections via its identification with the moduli space of stable holomorphic bundles.

In [5], we extend the proofs by Itoh [15] and Kobayashi [16] of their results for $M_\kappa^w(X,g)$ to prove that the L^2 metric $\mathbf{g}$ and integrable almost complex structure $\mathbf{J}$ on $\mathscr{M}_\mathfrak{t}^{\mathrm{sm}}$ define a *Kähler form* $\boldsymbol{\omega} = \mathbf{g}(\cdot, \mathbf{J}\cdot)$ on $\mathscr{M}_\mathfrak{t}^{\mathrm{sm}}$.

When g (and other geometric perturbation parameters in the SO(3) monopole equations) are *generic*, then $\mathscr{M}_\mathfrak{t}^{\mathrm{sm}} = \mathscr{M}_\mathfrak{t}^{*,0}$ but if g is Kähler and thus non-generic, this equality need not hold.

We address this issue in [5] and outline them in [4].

Critical points in space of SO(3) monopoles III

As in the analysis [12, Sections 6 and 7] by Hitchin, the function f in (7) is a *moment map* for the *circle action* on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$, that is,

$$df = \iota_{\xi}\omega \quad \text{on } \mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}},$$

where the vector field ξ on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$ is the generator of the S^1 action on $\mathscr{M}_{\mathfrak{t}}$ given by scalar multiplication on the sections Φ.

Because the fundamental 2-form ω is *non-degenerate*, then

- $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ is a *critical point of* $f : \mathscr{M}_{\mathfrak{t}} \to \mathbb{R}$ if and only if
- $[A, \Phi]$ is a *fixed point of the* S^1 *action on* $\mathscr{M}_{\mathfrak{t}}$.

From our previous work on SO(3) monopoles [6], we know that $[A, \Phi]$ is a fixed point of the S^1 action on $\mathscr{M}_{\mathfrak{t}}$ if and only if

Critical points in space of SO(3) monopoles IV

- (A, Φ) is a *reducible pair* with $\Phi \not\equiv 0$, equivalent to a *Seiberg–Witten monopole*, or
- $\Phi \equiv 0$, equivalent to a *projectively anti-self-dual connection*.

Thus, as a consequence of Theorem 1 and our results in [6, 7] that identify the fixed points of the S^1 action on $\mathscr{M}_{\mathfrak{t}}$, we have the

Theorem 5 (All critical points of Hitchin's Hamiltonian function represent either Seiberg–Witten monopoles or anti-self-dual connections)

Let $[A, \Phi] \in \mathscr{M}_{\mathfrak{t}}$ be a critical point of Hitchin's function (7). If $\Phi \not\equiv 0$, then there exists a spin^c structure $\mathfrak{s}$ over X such that $[A, \Phi] \in M_{\mathfrak{s}} \subset \mathscr{M}_{\mathfrak{t}}$.

Virtual Morse–Bott properties of critical points I

If a Seiberg–Witten fixed point $p = [A, \Phi]$ is a **smooth** point of $\mathscr{M}_\mathfrak{t}$ then, by arguments generalizing those of Hitchin [12, Section 7], one can apply Frankel's Theorem 1 to

- prove that f is Morse–Bott at p and
- compute the Morse index of f (the dimension of the maximal negative definite subspace of $\operatorname{Hess} f(p)$ on $T_p\mathscr{M}_\mathfrak{t}$) as the dimension of the negative weight space $T_p^-\mathscr{M}_\mathfrak{t}$ for the S^1 action on $T_p\mathscr{M}_\mathfrak{t}$.

The dimension of $T_p^-\mathscr{M}_\mathfrak{t}$ can be computed via the Atiyah–Singer Index Theorem or the Hirzebruch–Riemann–Roch Index Theorem via the identification of $\mathscr{M}_\mathfrak{t}$ with a moduli space of stable holomorphic pairs.

If the Seiberg–Witten fixed point $p = [A, \Phi]$ is a **singular** point of $\mathscr{M}_\mathfrak{t}$, as is more typical, we use the fact that $M_\mathfrak{s} \subset \mathscr{M}_\mathfrak{t}$ is a submanifold of a

Virtual Morse–Bott properties of critical points II

smooth virtual moduli space $\mathscr{M}_{\mathfrak{t}}^{\mathrm{vir}} \subset \mathscr{C}_{\mathfrak{t}}$ implied by the Kuranishi model given by the *elliptic deformation complex* for the SO(3) monopole equations with differentials $d_{A,\Phi}^{\bullet}$ and cohomology groups $\mathbf{H}_{A,\Phi}^{\bullet}$.

The *differentials* $d_{A,\Phi}^{\bullet}$ are given by the linearizations

1. $d_{A,\Phi}^{0}$ at id_E of the map $\mathscr{G}_E \ni u \mapsto u(A,\Phi) \in \tilde{\mathscr{C}}_{\mathfrak{t}}$;
2. $d_{A,\Phi}^{1}$ at (A,Φ) of SO(3) monopole map $\tilde{\mathscr{C}}_{\mathfrak{t}} \ni (A,\Phi) \mapsto \mathfrak{S}(A,\Phi)$.

Virtual Morse–Bott properties of critical points III

The *cohomology groups* $\mathbf{H}^{\bullet}_{A,\Phi}$ are given by

1. $\mathbf{H}^0_{A,\Phi} = \operatorname{Ker} d^0_{A,\Phi}$, *Lie algebra of isotropy subgroup* of $\mathscr{G}_E$ at (A,Φ);
2. $\mathbf{H}^1_{A,\Phi} = \operatorname{Ker}(d^1_{A,\Phi} + d^{0,*}_{A,\Phi})$, *Zariski tangent space* $T_{A,\Phi}\mathscr{M}_{\mathfrak{t}}$ and the *tangent space* $T_{A,\Phi}\mathscr{M}_{\mathfrak{t}}^{\mathrm{vir}}$;
3. $\mathbf{H}^2_{A,\Phi} = \operatorname{Ker} d^{1,*}_{A,\Phi}$, *obstruction to smoothness* at (A,Φ).

The space $\mathscr{M}_{\mathfrak{t}}^{\mathrm{vir}}$ is a complex Kähler manifold of dimension equal to that of the Zariski tangent space $T_p\mathscr{M}_{\mathfrak{t}}$ and contains $\mathscr{M}_{\mathfrak{t}}^{\mathrm{sm}}$ and $M_{\mathfrak{s}}$ as complex Kähler submanifolds.

The set of fixed points of the S^1 action on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{vir}}$ coincides with $M_{\mathfrak{s}}$ and f is Morse–Bott on $\mathscr{M}_{\mathfrak{t}}^{\mathrm{vir}}$ with critical submanifold $M_{\mathfrak{s}}$.

Virtual Morse–Bott properties of critical points IV

The moduli space of Seiberg–Witten monopoles $M_{\mathfrak{s}}$ is a smooth manifold but **not necessarily an embedded smooth submanifold** of $\mathscr{M}_{\mathfrak{t}}$.

The elliptic deformation complex defining $\mathbf{H}^{\bullet}_{A,\Phi}$ splits into a

- **normal** deformation subcomplex with cohomology groups $\mathbf{H}^{\bullet,n}_{A,\Phi}$, and a
- **tangential** deformation subcomplex with cohomology groups $\mathbf{H}^{\bullet,t}_{A,\Phi}$.

The normal elliptic deformation complex defining $\mathbf{H}^{\bullet,n}_{A,\Phi}$ further splits into a

- **positive weight** subcomplex with cohomology groups $\mathbf{H}^{\bullet,+}_{A,\Phi}$, and a
- **negative weight** subcomplex with cohomology groups $\mathbf{H}^{\bullet,-}_{A,\Phi}$.

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Virtual Morse–Bott properties of critical points V

We define the **virtual Morse–Bott index** of f at p to be minus the (Atiyah–Singer) index of the **negative weight, normal elliptic deformation subcomplex** defining $\mathbf{H}^{\bullet,-}_{A,\Phi}$ and compute these indices using the Hirzebruch–Riemann–Roch Index Theorem to give

Virtual Morse–Bott properties of critical points VI

Theorem 6 (Virtual Morse–Bott index of Hitchin's Hamiltonian function at a point represented by a Seiberg–Witten monopole)

Let X be a closed complex Kähler surface and $[A,\Phi] \in M_{\mathfrak{s}} \subset \mathscr{M}_{\mathfrak{t}}$ be a Seiberg–Witten monopole in the SO(3) *monopole moduli space $\mathscr{M}_{\mathfrak{t}}$. The virtual Morse–Bott index of Hitchin's function f in* (7) *at $[A,\Phi]$ is*

$$\begin{aligned}\lambda_{[A,\Phi]}(f) &= \dim \mathbf{H}^{1,-}_{A,\Phi} - \dim \mathbf{H}^{2,-}_{A,\Phi} - \dim \mathbf{H}^{0,-}_{A,\Phi} \\ &= -\chi_h(X) + \frac{1}{2}(c_1(\mathfrak{s}) - c_1(\mathfrak{t}))\cdot K_X - \frac{1}{2}(c_1(\mathfrak{s}) - c_1(\mathfrak{t}))^2 ,\end{aligned} \tag{8}$$

where $K_X \in H^2(X;\mathbb{Z})$ denotes the canonical class of X, and $c_1(\mathfrak{s}) := c_1(W^+) \in H^2(X;\mathbb{Z})$ for $\rho = (\mathfrak{s}, W)$ and $W = W^+ \oplus W$, and $c_1(\mathfrak{t}) := c_1(E) \in H^2(X;\mathbb{Z})$.

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Virtual Morse–Bott properties of critical points VII

We included $\mathbf{H}^{0,-}_{A,\Phi}$ above for completeness, but for SO(3) monopoles we have $\mathbf{H}^{0}_{A,\Phi} = 0$ unless $\Phi \equiv 0$ and A is reducible (a case that can be excluded by standard techniques for the moduli space of SO(3) monopoles.

The formula (8) for the virtual Morse–Bott index in the conclusion of Theorem 6 continues to hold for standard four-manifolds.

Thank you for your attention!

Appendix on unstable manifolds, resolution of singularities, and virtual Morse index

Unstable manifolds and resolution of singularities I

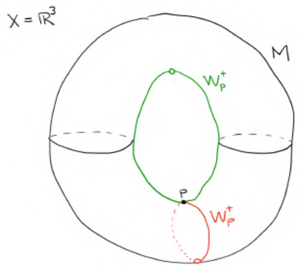

Figure 7.3: Stable and unstable manifolds $W^{\pm}(p)$ for the height function around a saddle point p in the torus in M in $X = \mathbb{R}^3$

Unstable manifolds and resolution of singularities II

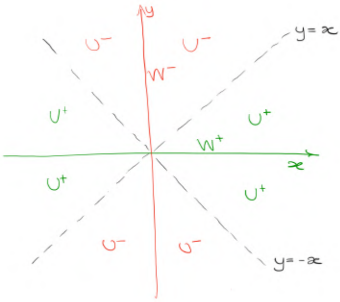

Figure 7.4: Open neighborhoods $U^{\pm}$ in $\mathbb{R}^2$ of the stable and unstable manifolds $W^{\pm}$ for the gradient flow of $f(x, y) = \frac{1}{2}(x^2 - y^2)$ near the origin

Unstable manifolds and resolution of singularities III

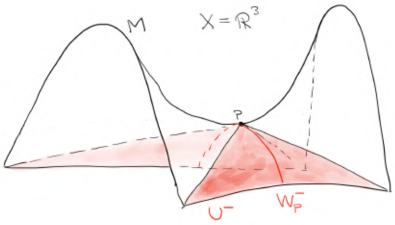

Figure 7.5: Open neighborhood U^{-} of the unstable manifold W_p^{-} for the gradient flow of the height function near a saddle point p in the torus M in $X = \mathbb{R}^3$

Unstable manifolds and resolution of singularities IV

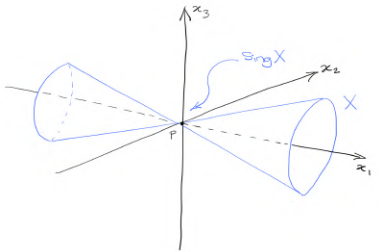

Figure 7.6: Quadratic cone $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 - x_2^2 - x_3^2 = 0\}$ in $M = \mathbb{R}^3$, singular locus $X_{\mathrm{sing}} = \{p\}$, and smooth function $f : M \to \mathbb{R}$ with $f : X \to \mathbb{R}$ assumed to have critical point at $p = (0,0,0)$, so $\operatorname{Ker} df(p) = T_pX = \operatorname{Ker} dF(p) = \mathbb{R}^3 = T_pM$ and $\operatorname{Coker} dF(p) = \mathbb{R}$

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Unstable manifolds and resolution of singularities V

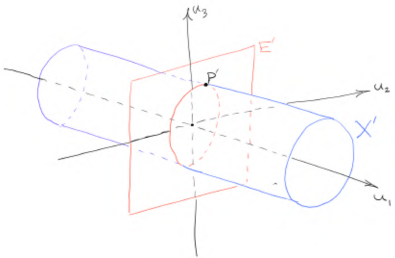

Figure 7.7: Resolution of the quadratic cone with strict transform $X' = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_2^2 + u_3^2 = 1\}$, exceptional divisor $E' = \{(u_1, u_2, u_3) \in \mathbb{R}^3 : u_1 = 0\}$, and point $p' \in E' \cap X'$ obtained by $x_1 = u_1$, $x_2 = u_1 u_2$, and $x_3 = u_1 u_3$

Unstable manifolds and resolution of singularities VI

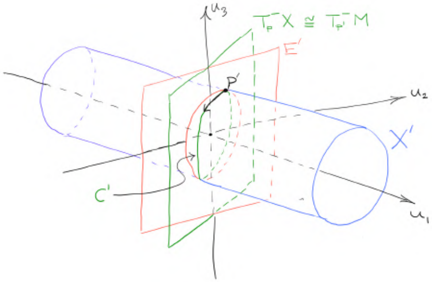

Figure 7.8: Resolution of the quadratic cone with strict transform X', exceptional divisor E', point $p' \in E' \cap X'$, two-dimensional subspace $T_p^- X \cong T_{p'}^- M$, one-dimensional unstable manifold $C' = X' \cap T_{p'}^- M$, and virtual Morse index $\lambda_p^-(f) := \dim T_p^- X - \dim \operatorname{Coker} dF(p) = 2 - 1 = 1$

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Unstable manifolds and resolution of singularities VII

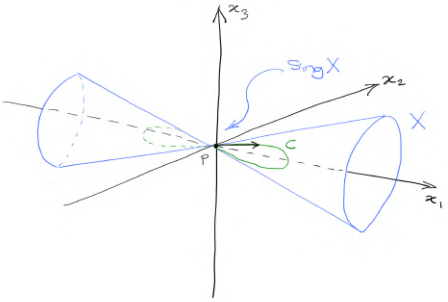

Figure 7.9: Quadratic cone with singular locus $X_{\text{sing}} = \{p\}$ and one-dimensional unstable subvariety $C = \pi(C')$

Unstable manifolds and resolution of singularities VIII

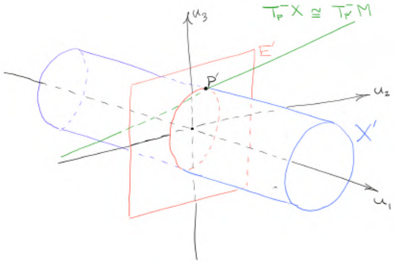

Figure 7.10: Resolution of the quadratic cone with strict transform X', exceptional divisor E', point $p' \in E' \cap X'$, and one-dimensional subspace $T_p^- X \cong T_{p'}^- M$, and virtual Morse index $\lambda_p^-(f) = \dim T_p^- X - \dim \operatorname{Coker} dF(p) = 1 - 1 = 0$

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