An introduction to KK-theory

Atiyah & Jänich showed that $K^0(X) = [X, \mathcal{F}]$

Atiyah defined an (abstract) elliptic operator on X to be a bdd Fredholm linear map $T: \mathcal{H}_0 \to \mathcal{H}_1$ between Hilbert spaces with an action of $C(X)$ s.t. $Tf - fT$ is compact for all $f \in C(X)$ and showed that $\mathrm{Ell}(X) \twoheadrightarrow K_0(X)$

Brown-Douglas-Fillmore & Kasparov figured out how to define $K_0(X)$ analytically

BDF came at this by classifying s.e.s. $0 \to \mathcal{K} \to E \to C(X) \to 0$ w/ E a C^*-alg

Kasparov tackled more general $0 \to \mathcal{K} \otimes B \to E \to A \to 0$ w/ A, B, E C^*-algs and so ended up defining a bivariant functor $KK(A, B)$

$KK(\mathbb{C}, C(X)) = K^0(X) =$ Fredholm operators parametrized by X

$KK(C(X), \mathbb{C}) = K_0(X) =$ Elliptic operators on X

$KK(C(X), C(Y)) =$ Elliptic operators on X parametrized by Y

Let's describe the typical elements of $KK(C(X), B)$ w/ X a closed mfd, B unital

- A B-vb over X refers to a locally trivial bdle over X whose fibers are fin gen proj. (right) B-modules, w/ B-linear transition functions

  e.g. $B = \mathbb{C}$, $\mathbb{C}$-vb's over X

  $B = C(Y)$, $\mathbb{C}$-vb's over $X \times Y$

  if $\tilde{X} \to X$ is the universal cover of X, $G = \pi_1 X$, $B = C^*_r G$

  then $\tilde{X} \times_G C^*_r G \to X$ is the universal $C^*_r G$-bdle over X

- Suppose E_0, E_1 are B-vb's over X, a B-elliptic operator $D: C^\infty(M; E_0) \to C^\infty(M; E_1)$ refers to a B-linear elliptic ΨDO

  Such an operator extends to a B-linear bdd map between suitable Sobolev spaces (Hilbert B-modules) $\mathcal{E}_0, \mathcal{E}_1$ and is B-Fredholm meaning that there is a decomposition $\mathcal{E}_0 = \mathcal{E}_0' \oplus \mathcal{E}_0''$, $\mathcal{E}_1 = \mathcal{E}_1' \oplus \mathcal{E}_1''$

$D: \mathcal{E}_0' \cong \mathcal{E}_1'$, $D: \mathcal{E}_0'' \to \mathcal{E}_1''$, $\mathcal{E}_0''$, $\mathcal{E}_1''$ f.g. proj.

This means that "up to a B-compact perturbation" Ker D, Cok D are f.g. proj.

The Mishchenko-Fomenko index of D is $\mathrm{Ind}(D) = [\mathcal{E}_0''] - [\mathcal{E}_1''] \in K_0(B)$

· A Hilbert B-module $\mathcal{E}$ is a right B-module equipped w/ a B-valued inner product

$\langle \cdot, \cdot \rangle$, right B-linear in the second variable, satisfying

$\langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*$, $\langle \xi, \xi \rangle \geq 0$ w/ equality iff $\xi = 0$

such that $\mathcal{E}$ is complete wrt the norm $\|\xi\|_{\mathcal{E}} = \| \langle \xi, \xi \rangle \|_B^{1/2}$

Associated to $\mathcal{E}$ are the adjointable operators

$\mathrm{End}_B^*(\mathcal{E}) = \{ T : \mathcal{E} \to \mathcal{E}$ bdd B-linear : $\exists\, T^*$ s.t. $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle \ \forall \xi, \eta \in \mathcal{E} \}$

& the B compact operators

$\mathcal{K}_B(\mathcal{E})$ = closure of the linear span of the "rk one operators" $T_{\xi,\eta}$

defined by $T_{\xi,\eta}(\nu) = \xi \langle \eta, \nu \rangle$

Rmk: Instead of $D: C^\infty(X; E_0) \to C^\infty(X; E_1)$ we work w/ $\mathbb{Z}_2$-graded spaces

$C^\infty(X; E_0 \oplus E_1)$ & replace D w/ $\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$

Also note that if D is s.a. & has order one then $T = D(1+D^2)^{-1/2}$

has order zero and satisfies $T^2 - I = -(1+D^2)^{-1} \in \mathcal{K}_B(\mathcal{E})$

A class in $KK(A,B)$ is represented by a Kasparov A-B bimodule $(\mathcal{E}, \phi, T)$

where $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ is a $\mathbb{Z}_2$-graded Hilbert B-module

ϕ is a $*$-hom $A \to \mathrm{End}_B^*(\mathcal{E})$ & is even wrt the grading

& $T \in \mathrm{End}_B^*(\mathcal{E})$ is odd wrt the grading and satisfies, for any $a \in A$,

$\phi(a)(T^2 - 1)$, $\phi(a)(T - T^*)$, $\phi(a)T - T\phi(a) \in \mathcal{K}_B(\mathcal{E})$

Two Kasparov A-B bimodules $(\mathcal{E}_0, \phi_0, T_0)$, $(\mathcal{E}_1, \phi_1, T_1)$ are equivalent iff homotopic

meaning that there is a Kasparov $A - B \otimes C([0,1])$ bimodule $(\mathcal{E}, \phi, T)$

s.t. $(\mathcal{E}_i, \phi_i, T_i) \simeq (\mathcal{E} \otimes_{ev_i} B, \phi_i \otimes 1, T_i \otimes 1)$ $\quad i \in \{0,1\}$

$KK(A,B)$ consists of the equivalence classes & is a group wrt direct sum

($KK^1(A,B)$ is defined in the same way, but using ungraded objects)

Ex: ·) any homomorphism $\phi: A \to B$ yields a class $[\phi] \in KK(A,B)$

we take $\mathcal{E} = B \oplus 0$, $\langle b_1, b_2 \rangle_{\mathcal{E}} = b_1^* b_2$, $T = 0$

Here $End_B^*(\mathcal{E}) = M(B)$ & $\mathcal{K}_B(\mathcal{E}) = B$, so $\phi(A) \subseteq \mathcal{K}_B(\mathcal{E})$

For $id: A \to A$ this class is denoted $1_A \in KK(A,A)$

·) $D \in Diff^1(X; E_0, E_1)$ B-elliptic

$\mathcal{E} = L^2(X; E_0) \oplus L^2(X; E_1)$, $A = C(X)$, $T = D(1 + D^*D)^{-1/2}$

yields an element of $KK(C(X), B)$

Kasparov product: functorial, bilinear, associative product $KK(A,B) \times KK(B,C) \xrightarrow{*} KK(A,C)$

Ex: Suppose $D \in Diff^1(X; E_0, E_1)$ & $F \to X$ vb w/ connection ∇^F

$\exists!$ $D' \in Diff^1(X; E_0 \otimes F, E_1 \otimes F)$ s.t. $D'(s \otimes z)(x) = (Ds \otimes z)(x)$

whenever $\nabla^F z(x) = 0$

(if $D = cl \circ \nabla^E$ then $D' = cl \circ \nabla^{E \otimes F}$)

D defines a class via $T = D(1 + D^*D)^{-1/2}$ in $KK(C(X), \mathbb{C})$

& similarly for D'

F defines a class $[F] \in KK(\mathbb{C}, C(X))$ $\qquad [C_0(F) \oplus 0, \mu, 0]$

& defines a class $[[F]] \in KK(C(X), C(X))$

$[D'] = [[F]] * [D]$ $\qquad$ note that $[D']$ is independent of ∇^F

$\qquad A - B \qquad\qquad B - C \qquad\qquad A - C$

Given $(\mathcal{E}_1, \phi_1, T_1)$ & $(\mathcal{E}_2, \phi_2, T_2)$ we want $(\mathcal{E}, \phi, T)$

Clearly we should take $\mathcal{E} = \mathcal{E}_1 \otimes_B \mathcal{E}_2$, $\phi = \phi_1 \otimes 1 : A \to End_C^*(\mathcal{E})$

but there's no canonical choice for T

Rmk: The naive choice for T would be $T_1 \otimes 1 + \gamma_1 \otimes T_2$ w/ γ_1 the grading on $\mathcal{E}_1$

but usually $\gamma_1 \otimes T_2$ is not well-defined because $[T_2, B] \neq 0$

and even when it is it might not define a Kasparov bimodule

(modify to $M(T_1 \otimes 1) + N(\gamma_1 \otimes T_2)$ w/ $M, N \in \mathrm{End}^*_c(\mathcal{E})$, $M, N \geq 0$, $M^2 + N^2 = 1$

Kasparov's Technical Lemma)

Connes-Skandalis have characterized T (up to homotopy) using an abstract notion of connection inspired by the previous example

Ex: Thom isomorphism in K-thy

If $E \to X$ is a $\mathbb{C}$-vb (or a $\mathbb{R}$-vb w/ a spin^c-structure)

then $K^*(E) \simeq K^*(X)$

We obtain a class $\alpha_E \in KK(C_0(E), C_0(X))$ from the family of Dolbeault operators on the fibers of E, $\bar\partial_{E/X}$: $\alpha_E = [L^2(\Lambda^{0,*} E_x), \phi, \bar\partial_{E/X} + \bar\partial^*_{E/X}]$

Prop (Kasparov following Atiyah)

Multiplication by α_E is invertible and implements the Thom isomorphism

If X is itself complex then the classes of $\bar\partial_X$, $\bar\partial_E$, and $\bar\partial_{E/X}$ satisfy

$[\bar\partial_E] = \alpha_E \,\#\, [\bar\partial_X]$

Ex: Poincaré Duality

Atiyah showed that for a smooth mfd $K_0(X) \simeq K^0(T^*X)$

& that for $P \in \Psi^0(X; E_0, E_1)$ this correspondence sends $[P]$ to $[\sigma(P)]$

He pointed out that this generalizes the Atiyah-Singer index thm

Kasparov showed that this too is an Kasparov product

Note that $\sigma(P)$ defines a class $[\sigma(P)] \in KK(\mathbb{C}, C_0(T^*X))$

& also a class $[[\sigma(P)]] \in KK(C(X), C_0(T^*X))$

Thm (Kasparov) $[P] = [[\sigma(P)]] \otimes [\bar{\partial}_{T^*X}]$

Let $u : \mathbb{C} \hookrightarrow C(X)$ be the inclusion of units $[u] \in KK(\mathbb{C}, C(X))$

We get the analytic index of P $[u] \otimes [P] \in KK(\mathbb{C},\mathbb{C}) = \mathbb{Z}$

Kasparov's thm implies that this equals $[u] \otimes [[\sigma(P)]] \otimes [\bar{\partial}_{T^*X}]$

$= [\sigma(P)] \otimes [\bar{\partial}_{T^*X}]$

Ex: Topological index

Choose an embedding $X \hookrightarrow \mathbb{R}^n$ & let N be a tubular nbhd

so $X \hookrightarrow N \hookrightarrow \mathbb{R}^n$, N can be identified with the total space of the normal bdle

We get $TX \hookrightarrow TN \hookrightarrow T\mathbb{R}^n = \mathbb{C}^n$ & TN is a $\mathbb{C}$-vb over TX

Since $TN \hookrightarrow \mathbb{C}^n$ is an open inclusion it induces $j : C_0(TN) \hookrightarrow C_0(\mathbb{C}^n)$

The topological index is the map

$$K(T^*X) = K(TX) \xrightarrow{\otimes \alpha_{TN}^{-1}} K(TN) \xrightarrow{\otimes [j]} K(\mathbb{C}^n) \xrightarrow{\otimes [\bar{\partial}_{\mathbb{C}^n}]} K(pt) = \mathbb{Z}$$

Thm (Atiyah–Singer) a-ind = t-ind

Pf

From

$$\begin{tikzcd} K^0(TX) \arrow[r, "\otimes \alpha_{TN}^{-1}"] \arrow[dr, "\otimes[\bar{\partial}_{TX}]"'] & K(TN) \arrow[r, "\otimes [j]"] \arrow[d, "\otimes[\bar{\partial}_{TN}]"] & K(\mathbb{C}^n) \arrow[dl, "\otimes[\bar{\partial}_{\mathbb{C}^n}]"] \\ & K(pt) & \end{tikzcd}$$

we see that t-ind$(c) = c \otimes [\bar{\partial}_{TX}]$

$=$ a-ind(c) //