The Baum-Connes conjecture and quantum groups

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KK-theory and homotopy theory
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The Baum-Connes conjecture

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Let G be a second countable locally compact group. The *Baum-Connes conjecture* asserts that the assembly map

$$\mu : K_*^{top}(G) \to K_*(C_r^*(G))$$

is an isomorphism.

Here

$$K_*^{top}(G) = K_*^G(\mathcal{E}G) = \varinjlim_{X \subset \mathcal{E}G,\, X\ G\text{-compact}} KK_*^G(C_0(X), \mathbb{C})$$

is the equivariant K-homology of the universal proper G-space.

The conjecture is true in many cases - no counterexample is known.

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What happens if G is a locally compact *quantum* group?

The framework

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Meyer and Nest reformulated the Baum-Connes conjecture using the language of triangulated categories and derived functors (2004, 2006). This yields

- a better understanding of the (classical) conjecture
- a framework to define and study assembly maps in other situations

In fact, one of the motivations for their work was to extend the Baum-Connes machinery to the realm of quantum groups.

Based on their approach, Meyer and Nest formulated and proved an analogue of the Baum-Connes conjecture for duals of compact groups (2007).

Equivariant Kasparov theory

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Let G be a locally compact group.

Equivariant Kasparov theory defines an additive category KK^G, with

- objects all separable G-C^*-algebras
- morphism sets the bivariant Kasparov K-groups $KK^G(A,B)$
- the composition of morphisms

  $$KK^G(A,B) \times KK^G(B,C) \to KK^G(A,C)$$

  given by *Kasparov product*.

Structure as a triangulated category

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The category KK^G is triangulated - this allows one to do homological algebra.

A triangulated category is an additive category togther with a translation functor and a class of exact triangles satisfying certain axioms.

In the case of KK^G, we have that

- the (inverse of the) *suspension* $\Sigma A = C_0(\mathbb{R}) \otimes A$ yields the translation functor.
- the exact triangles are all diagrams in KK^G isomorphic to mapping cone triangles
$$\Sigma B \to C_f \to A \to B$$
for equivariant $*$-homomorphisms $f : A \to B$.

Every extension $0 \to I \to A \to B \to 0$ of G-C^*-algebras with a G-equivariant completely positive contractive linear splitting defines an exact triangle.

Compactly induced and compactly contractible algebras

Compactly induced and compactly contractible algebras

A G-C^*-algebra is called *compactly induced* if it is of the form $\mathrm{ind}_H^G(B)$ for an H-C^*-algebra B and some compact subgroup $H \subset G$.

Here the induced G-C^*-algebra is defined as

$$\mathrm{ind}_H^G(B) = \{f \in C_0(G, B) \mid f(th) = h \cdot f(t) \text{ for all } h \in H\} \subset C_0(G, B),$$

equipped with the G-action $(g \cdot f)(t) = f(g^{-1}t)$.

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A G-C^*-algebra A is called *compactly contractible* if $\mathrm{res}_H^G(A) \cong 0 \in KK^H$ for every compact subgroup $H \subset G$.

We write $\langle \mathcal{CI} \rangle$ for the *localising subcategory* of KK^G generated by all compactly induced algebras and $\mathcal{CC}$ for the full subcategory of all compactly contractible algebras. The category $\mathcal{CC}$ is automatically localising.

Dirac triangles

Dirac triangles

Let G be a locally compact group.

- A morphism $f : A \to B$ in KK^G is a *weak equivalence* if $\mathrm{res}_H^G(f)$ is an isomorphism in $KK^H(A, B)$ for all compact subgroups $H \subset G$.
- A $\langle \mathcal{CI} \rangle$-simplicial approximation of a G-C^*-algebra A is a weak equivalence $\tilde{A} \to A$ with $\tilde{A} \in \langle \mathcal{CI} \rangle$.

Theorem (Meyer-Nest 2006)

For every $A \in KK^G$ there exists a $\langle \mathcal{CI} \rangle$-simplicial approximation $\tilde{A}$, unique up to isomorphism. This fits into an exact triangle

$$\Sigma N \to \tilde{A} \to A \to N,$$

called Dirac triangle, with $N \in \mathcal{CC}$.

Definition

Let A be a G-C^*-algebra. Then G satisfy the Baum-Connes conjecture with coefficients in A if the map

$$K_*(G \ltimes_r \tilde{A}) \to K_*(G \ltimes_r A)$$

is an isomorphism.

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Theorem (Higson-Kasparov 2001)

If G has the Haagerup property (is a-T-menable) then G satisfies the strong Baum-Connes conjecture.

What is... a quantum group?

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The term "quantum group" is typically used for quantized universal enveloping algebras. These originate from the study of the quantum inverse scattering method developed in the 1980's (Faddev-Reshetikhin-Takhtajan, Drinfeld-Jimbo and others).

In this context, quantum groups are certain (classes of) Hopf algebras.

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In this context, quantum groups are certain (classes of) Hopf algebras.

Independently, at around the same time, Woronowicz introduced the quantum group $SU_q(2)$ and developed the theory of (what is now called) compact quantum groups.

The general definition of a *locally compact* quantum group was later given by Kustermans and Vaes.

Pontrjagin duality

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Example

The compact group $G = S^1$ is Pontrjagin dual to the discrete group $\widehat{G} = \mathbb{Z}$. Using C^*-algebras this can be expressed via the isomorphisms

$$C(S^1) = C^*(\mathbb{Z}), \qquad C^*(S^1) \cong C_0(\mathbb{Z}),$$

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In the spirit of noncommutative topology, if G is a (possibly nonabelian) locally compact group, the correct replacement for the pair of Pontrjagin dual (quantum) groups should be $C_0(G)$ and $C^*(G)$.

Locally compact quantum groups

- *Duality for compact groups* - Tannaka (1938)
- *Kac algebras* - Kac-Vainerman, Enock-Schwartz (1973)
- $SU_q(2)$ and compact quantum groups - Woronowicz (1987)
- *Examples of and constructions with locally compact quantum groups* - Woronowicz and others (since 1990)

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Definition (Kustermans-Vaes 1999)

A *locally compact quantum group* is a C^*-algebra H together with a comultiplication $\Delta : H \to M(H \otimes H)$ and left and right Haar integrals.

Compact quantum groups

Compact quantum groups

Definition (Woronowicz)

A compact quantum group is given by a unital C^*-algebra S together with a unital $*$-homomorphism $\Delta : S \to S \otimes S$ such that

$$\begin{array}{ccc} S & \xrightarrow{\Delta} & S \otimes S \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \mathrm{id} \otimes \Delta} \\ S \otimes S & \xrightarrow{\Delta \otimes \mathrm{id}} & S \otimes S \otimes S \end{array}$$

is commutative and

$$\Delta(S)(1 \otimes S), \qquad (S \otimes 1)\Delta(S)$$

are dense subspaces of $S \otimes S$.

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is commutative and

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are dense subspaces of $S \otimes S$.

Every compact quantum group has a unique state ϕ, called *Haar integral*, satisfying

$$(\phi \otimes \mathrm{id})\Delta(x) = \phi(x)1 = (\mathrm{id} \otimes \phi)\Delta(x).$$

Example: Compact groups

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- If G is a compact group then $S = C(G)$ is a compact quantum group.

  The comultiplication $\Delta : C(G) \to C(G) \otimes C(G) = C(G \times G)$ is given by

  $$\Delta(f)(s,t) = f(st).$$

  The Haar integral is given by integration with respect to (normalised) Haar measure.

- Every compact quantum group for which S is a commutative C^*-algebra is of this form.

Example: Discrete groups

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- If G is a discrete group then $S = C^*_{\mathsf{r}}(G)$ is a compact quantum group.

  The comultiplication $\Delta : C^*_{\mathsf{r}}(G) \to C^*_{\mathsf{r}}(G) \otimes C^*_{\mathsf{r}}(G)$ is given by

  $$\Delta(s) = s \otimes s$$

  for $s \in G \subset \mathbb{C}G \subset C^*_{\mathsf{r}}(G)$.

  The Haar integral is given by

  $$\phi(x) = \langle \delta_e, x\delta_e \rangle$$

  for $x \in C^*_{\mathsf{r}}(G) \subset B^2(L^2(G))$.

- In a similar way $S = C^*_{\mathsf{f}}(G)$ is a compact quantum group.

- Every compact quantum group which is *cocommutative* is (essentially) of this form.

Example: The quantum group $SU_q(2)$

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Fix $q \in [-1, 1] \setminus \{0\}$.

Definition (Woronowicz 1987)

The C^*-algebra $C(SU_q(2))$ is the universal C^*-algebra generated by elements α and γ satisfying the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

These relations are equivalent to saying that the fundamental matrix

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary.

Example: The quantum group $SU_q(2)$

The comultiplication $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ is defined by

$$\Delta \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

using “matrix multiplication”, that is,

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma$$

and similarly for the other generators.

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For $q = 1$ one (re-)obtains in this way the C^*-algebra $C(SU(2))$ of functions on $SU(2)$ together with the group structure of $SU(2)$.

Example: Free orthogonal quantum groups

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Definition (Van Daele-Wang 1995)

Let $n \in \mathbb{N}$. The C^*-algebra $C^*_{\mathrm{f}}(\mathbb{F}O(n)) = A_o(n)$ of the free orthogonal quantum group $\mathbb{F}O(n)$ is the universal C^*-algebra with self-adjoint generators $u_{ij}, 1 \leq i, j \leq n$ and relations

$$\sum_{k=1}^{n} u_{ik} u_{jk} = \delta_{ij}, \qquad \sum_{k=1}^{n} u_{ki} u_{kj} = \delta_{ij}.$$

- These relations are equivalent to saying that $u = (u_{ij})$ is an orthogonal matrix.
- The *abelianisation* of $C^*_{\mathrm{f}}(\mathbb{F}O(n))$ is isomorphic to the algebra $C(O(n))$ of functions on the orthogonal group $O(n)$.

The *comultiplication* $\Delta : C^*_{\mathrm{f}}(\mathbb{F}O(n)) \to C^*_{\mathrm{f}}(\mathbb{F}O(n)) \otimes C^*_{\mathrm{f}}(\mathbb{F}O(n))$ is defined by

$$\Delta \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix} \otimes \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}.$$

Explicitly,

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Free orthogonal quantum groups and nonamenability

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The reduced C^*-algebra $C^*_{\mathsf{r}}(\mathbb{F}O(n))$ is the image of $C^*_{\mathsf{f}}(\mathbb{F}O(n))$ in the GNS-representation of the Haar integral.

For $n > 2$ the free orthogonal quantum group $\mathbb{F}O(n)$ is not amenable, that is, the canonical map

$$\lambda : C^*_{\mathsf{f}}(\mathbb{F}O(n)) \to C^*_{\mathsf{r}}(\mathbb{F}O(n))$$

is *not* an isomorphism.

From non-amenability it follows that $C^*_{\mathsf{f}}(\mathbb{F}O(n))$ and $C^*_{\mathsf{r}}(\mathbb{F}O(n))$ are not nuclear.

The tensor category of representations

Definition

Let G be a compact quantum group and $S = C(G)$. A unitary representation of G on a Hilbert space $\mathcal{H}$ is a unitary $U \in B(S \otimes \mathcal{H})$ such that

$$(\Delta \otimes \mathrm{id})(U) = U_{13} U_{23}.$$

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- There is an obvious way to define the *direct sum* and the *tensor product* of representations.
- A representation is irreducible if it cannot be written as the direct sum of two representations.
- Every irreducible representation of a compact quantum group is finite dimensional.
- The finite dimensional representations of the compact quantum group G form a C^*-tensor category $\mathrm{Rep}(G)$.

Meyer-Nest for quantum groups

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In order to formulate an analogue of the Baum-Connes conjecture one also needs a suitable choice of localising subcategories $\langle \mathcal{CI} \rangle$ and $\mathcal{CC}$.

The notion of a compact quantum subgroup, together with restriction and induction, makes sense - but basic examples show that this is not quite the right thing to look at in general...

For discrete quantum groups, a "good" choice of $\langle \mathcal{CI} \rangle$ and $\mathcal{CC}$ have been proposed by Arano-Skalski (2021).

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For given (discrete) G one can also just *try* choices of $\mathcal{CC}$ and $\langle \mathcal{CI} \rangle$ and see what the corresponding assembly map gives!

If G is *torsion-free* then a natural choice of *compactly induced G-C^*-algebras* are those of the form $C_0(G) \otimes A$ where A is any C^*-algebra.

In this case a G-C^*-algebra A is *compactly contractible* if $A \cong 0$ in KK.

For free orthogonal quantum groups the above choice of $\mathcal{CC}$ and $\langle \mathcal{CI} \rangle$ works.

Theorem (V. 2009)

The free orthogonal quantum group $\mathbb{F}O(n)$ satisfies the strong Baum-Connes conjecture.

Baum-Connes for free orthogonal quantum groups

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Theorem (V. 2009)

The free orthogonal quantum group $\mathbb{F}O(n)$ satisfies the strong Baum-Connes conjecture.

Corollary

- *The free orthogonal quantum group $\mathbb{F}O(n)$ is K-amenable.*
- *In particular, the natural map*

$$K_*(C^*_{\mathsf{f}}(\mathbb{F}O(n))) \to K_*(C^*_{\mathsf{r}}(\mathbb{F}O(n)))$$

  is an isomorphism.
- *The K-theory of $\mathbb{F}O(n)$ is given by*

$$K_0(C^*(\mathbb{F}O(n))) = \mathbb{Z}$$
$$K_1(C^*(\mathbb{F}O(n))) = \mathbb{Z}.$$

  for all $n > 2$.

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- Applying maximal resp. reduced crossed products to D yields a commutative diagram

$$\begin{array}{ccc} G \ltimes_{\mathsf{f}} \tilde{\mathbb{C}} & \longrightarrow & G \ltimes_{\mathsf{f}} \mathbb{C} = C^*_{\mathsf{f}}(G) \\ \downarrow & & \downarrow \\ G \ltimes_{\mathsf{r}} \tilde{\mathbb{C}} & \longrightarrow & G \ltimes_{\mathsf{r}} \mathbb{C} = C^*_{\mathsf{r}}(G) \end{array}$$

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  in KK.
- The left vertical arrow is *always* an isomorphism.
- *Strong BC implies that D is an isomorphism.*
- Hence the horizontal arrows are isomorphisms.
- So the right arrow is an isomorphism in KK as well, and and hence induces an isomorphism in K-theory.

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- Using a Koszul resolution argument one obtains an exact triangle

$$C_0(G) \longrightarrow \tilde{\mathbb{C}} \longrightarrow \Sigma C_0(G) \longrightarrow \Sigma C_0(G)$$

which, after taking crossed products, induces an exact sequence

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(G \ltimes \tilde{\mathbb{C}}) & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & K_1(G \ltimes \tilde{\mathbb{C}}) & \longleftarrow & 0 \end{array}$$

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- Using the counit $\epsilon : C^*_{\mathsf{f}}(G) \to \mathbb{C}$ we see that $K_0(G \ltimes \tilde{\mathbb{C}}) \cong K_0(C^*_{\mathsf{f}}(G))$ contains a direct summand $\mathbb{Z}$.

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- The crossed product $G \ltimes C_0(G)$ is isomorphic to the compact operators $\mathbb{K}$.
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- ...this yields the claim.

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The strong Baum-Connes property does not depend on $G = \mathbb{F}O(n)$ but only on the tensor category $\mathrm{Rep}(G)$.

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In fact, Arano-Kitamura-Kubota develop equivariant KK-theory for actions of C^*-tensor categories on C^*-algebras.

Since Rep$(\mathbb{F}O(n)) \cong$ Rep$(SU_q(2))$ for $n = -q - q^{-1}$ it suffices to show

Theorem (V. 2009)

The dual of $SU_q(2)$ satisfies the strong Baum-Connes conjecture.

The proof strategy for this is similar to the proof of Baum-Connes for the classical group $SL(2, \mathbb{C})$.