r-spin structures and applications

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Represents joint work with
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February 16, 2021

r-spin structures

An r-spin structure is a…

C smooth algebraic curve, K_C canonical (cotangent) bundle.

line bundle L with $rL = K_C$

vector field with zeroes of order kr up to isotopy

$r = 0$: *framing* of S

S topological surface, UTS unit tangent bundle.

$\mathbb{Z}/r\mathbb{Z}$ cover $\widetilde{UTS} \to UTS$ restricting to a *connected* cover $S^1 \to S^1$

$\mathcal{S}(S)$: set of oriented SCC's / isotopy.

function $\phi : \mathcal{S}(S) \to \mathbb{Z}/r\mathbb{Z}$ satisfying certain axioms "Winding number function"

Examples

- $r = 2$: all over classical algebraic geometry

  (special divisors on algebraic curves, e.g. Weierstrass points, bitangents)

- Smooth plane curves (e.g. $X^d + Y^d + Z^d = 0$):

  Adjunction formula: $K_C = \mathcal{O}(d-3)|_C$.

  Implies $\mathcal{O}(1)|_C$ is a $d-3$- spin structure.

- Generalizes to any algebraic surface with $Pic(X)$ discrete, torsion-free.

- $f : \mathbb{C}^2 \to \mathbb{C}$ isolated plane curve singularity:

  Milnor fibers carry a "Gelfand-Leray form" $\dfrac{dX}{f_Y}$

  Nowhere-vanishing: $r = 0$ (framing)

- (C, ω) *translation surface*

  $C \backslash Z(\omega)$ carries framing induced by ω ("horizontal vector field")

Monodromy

Common setup: $p : \mathscr{X} \to B$ family (bundle) of Riemann surfaces

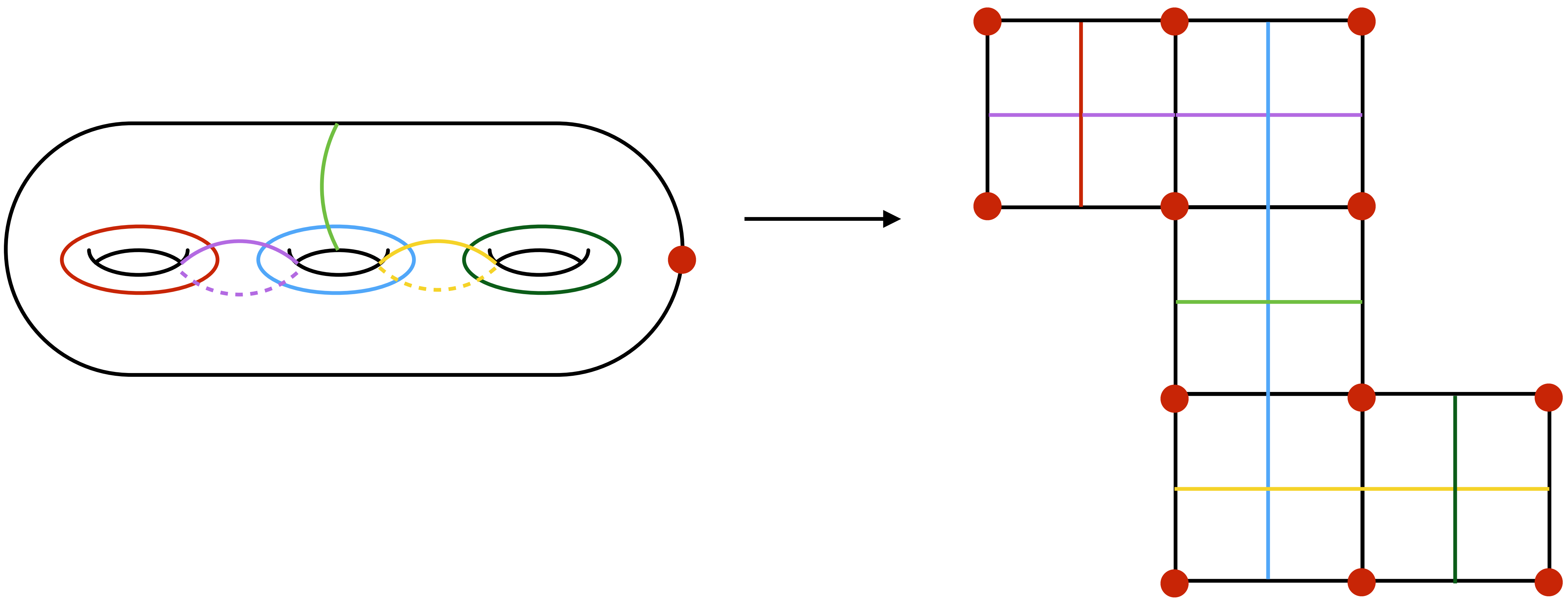
Topology (and more!) governed by *monodromy:*

$$\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma)$$

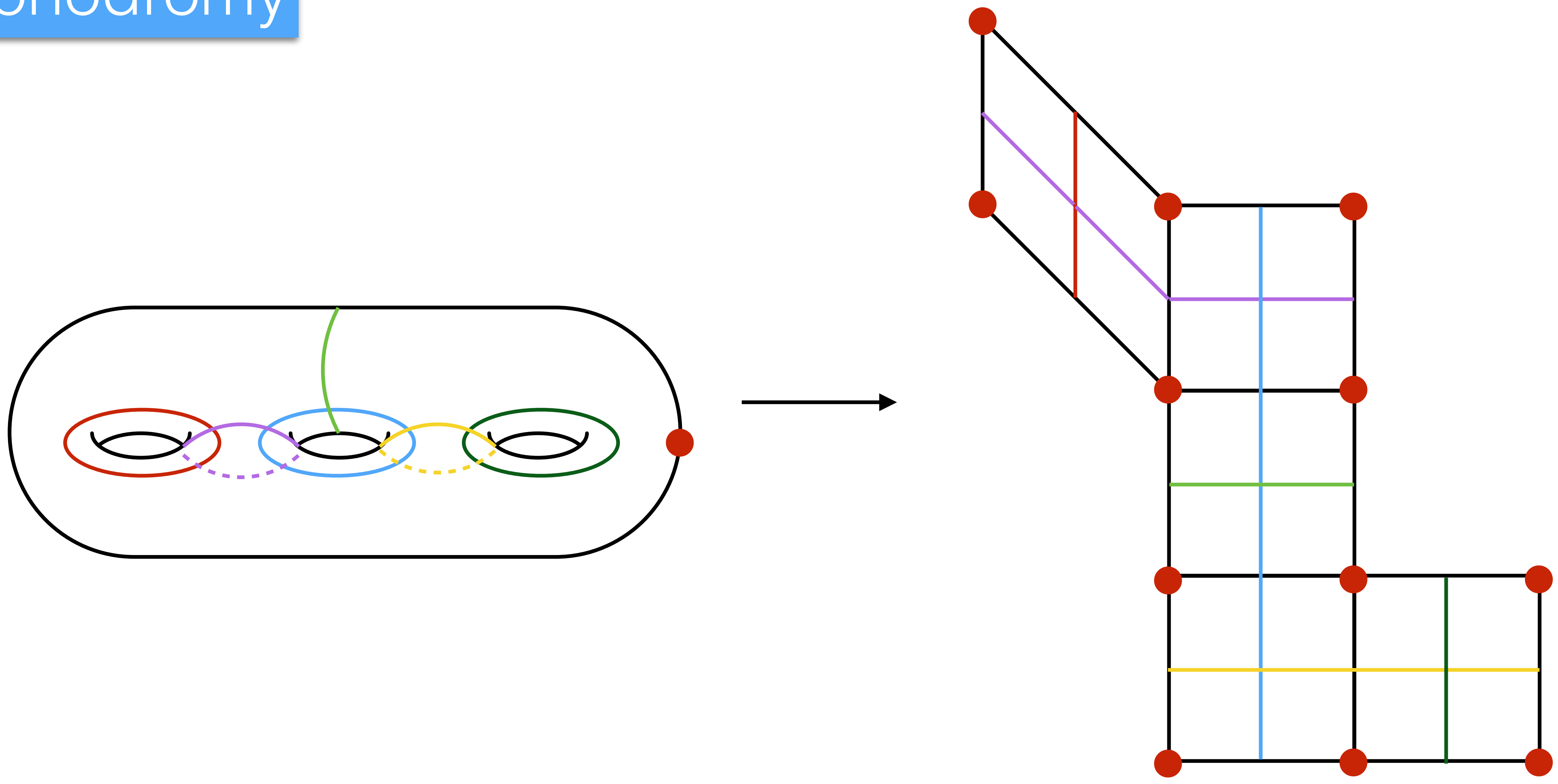
Recall $\mathrm{Mod}(\Sigma)$ is the *mapping class group:* diffeos up to isotopy.

Given a loop in B, monodromy describes "twisting" of fiber when pushed all the way around.

Monodromy



Monodromy



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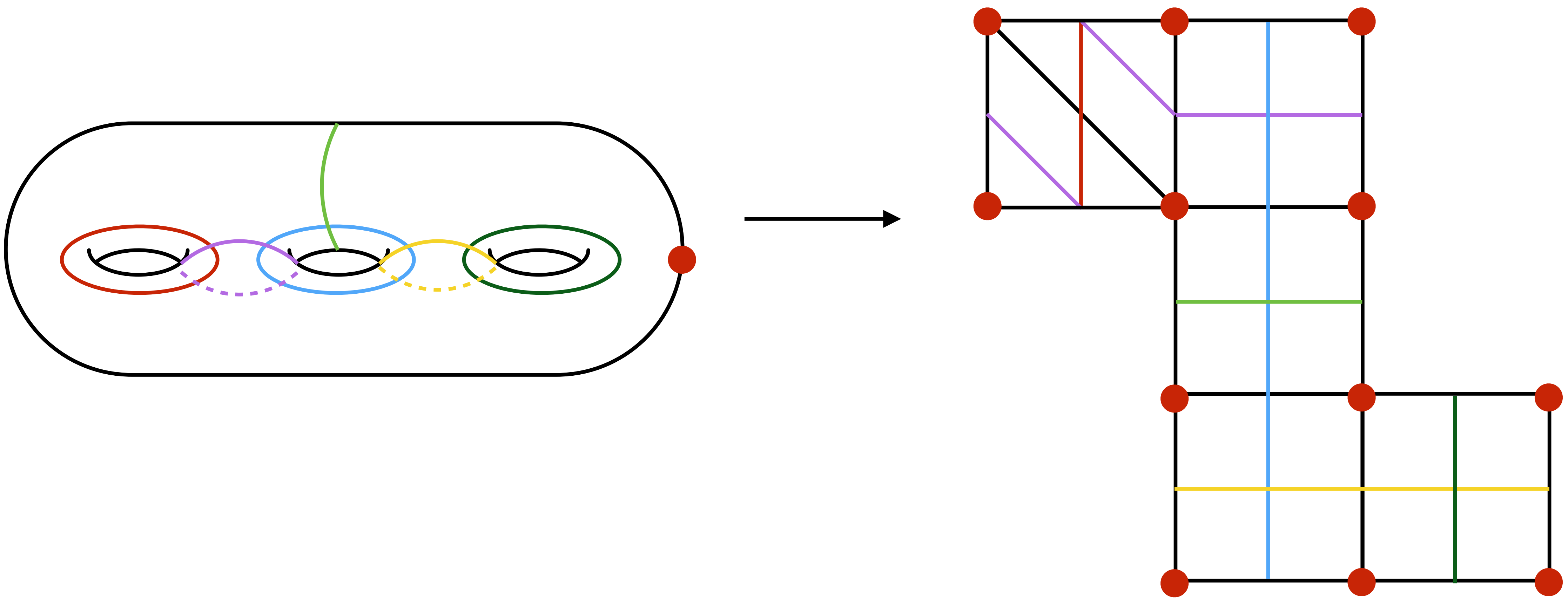

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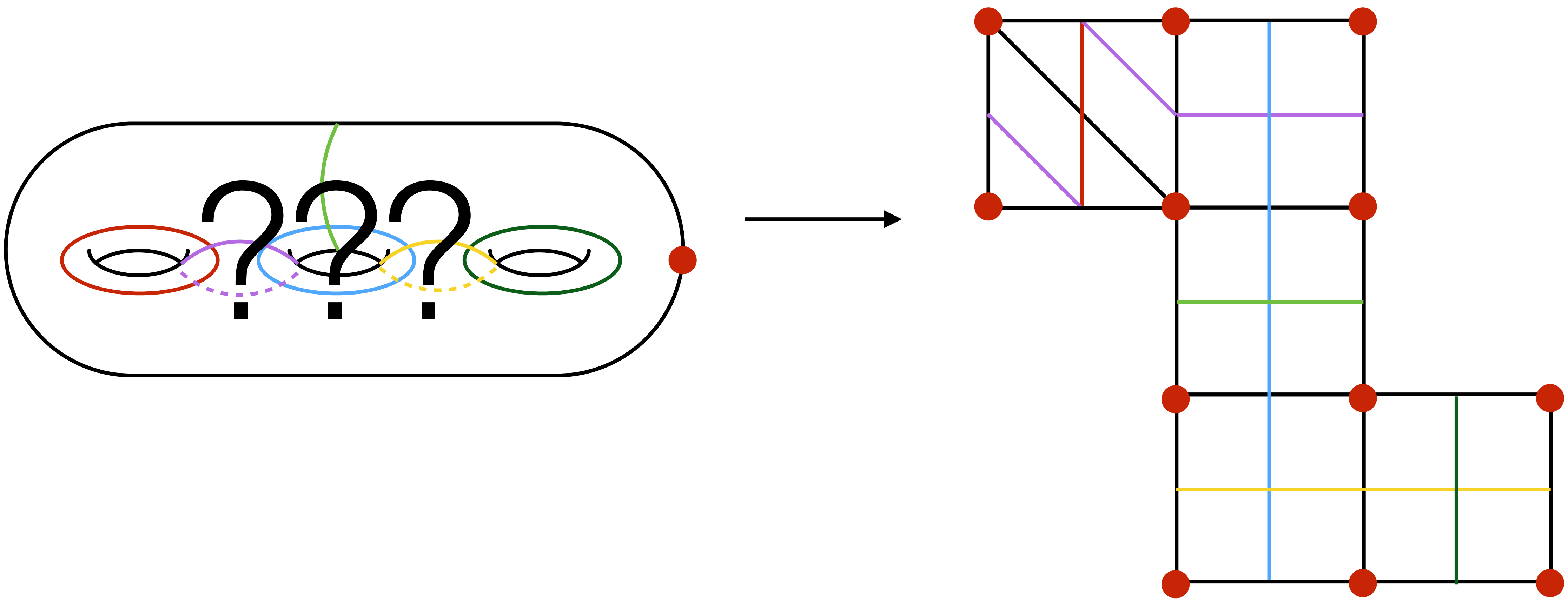

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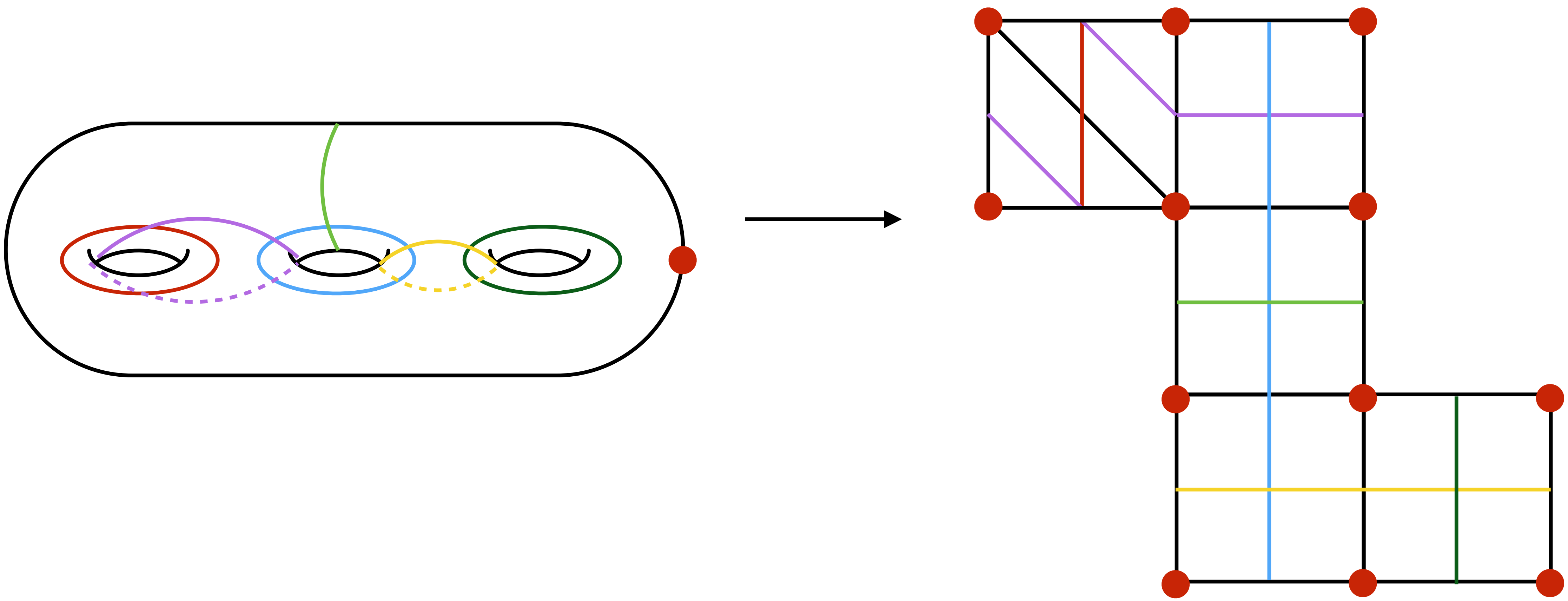

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Basic problem: describe (image of) ρ

Key observation: ρ *detects* presence of r-spin structure

r-spin mapping class groups

There is *action* $\mathrm{Mod}(\Sigma) \circlearrowright \{rSS\}$

Define $\mathrm{Mod}(\Sigma)[\phi]$ as the stabilizer of $\phi \in \{rSS\}$.

In the presence of a canonical r-spin structure:

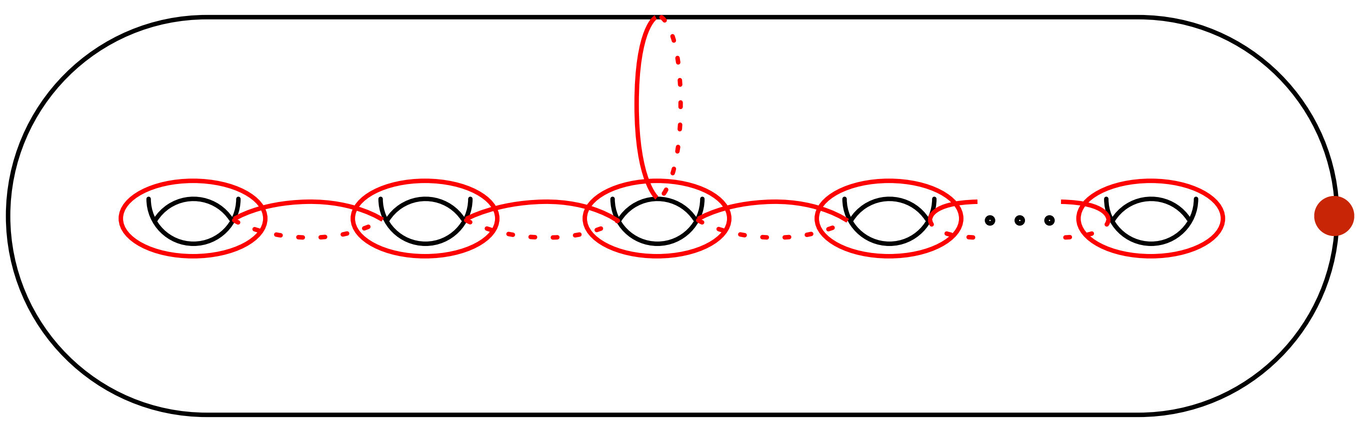
$$\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma)[\phi]$$

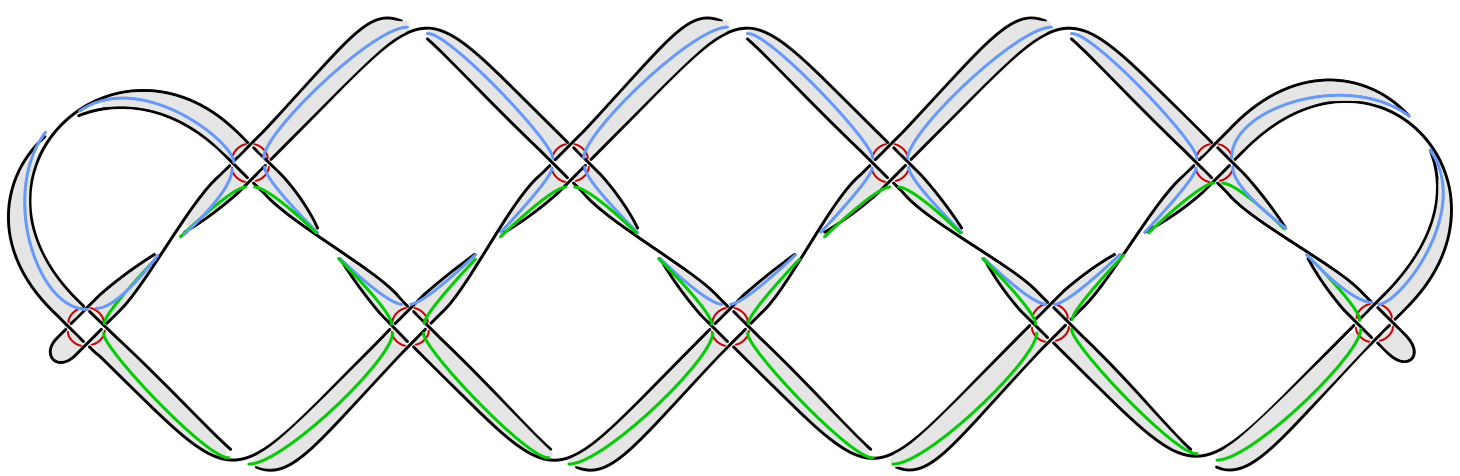
To understand monodromy, need technology to show that ρ *surjects.*

Need *generators* for $\mathrm{Mod}(\Sigma)[\phi]$

Main theorem

Theorem (Calderon - S.): For g ≥ 5, any framing ϕ, $\mathrm{Mod}(\Sigma)[\phi]$ is generated by finitely many Dehn twists.



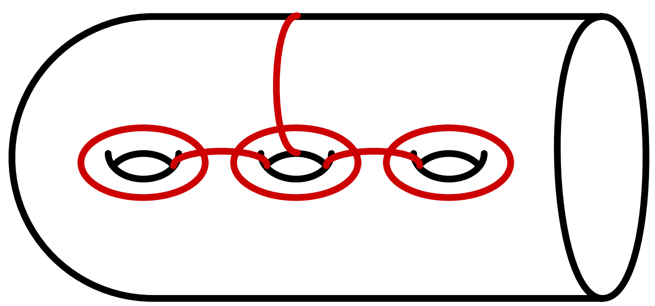


Even though $[\mathrm{Mod}(\Sigma) : \mathrm{Mod}(\Sigma)[\phi]] = \infty$!

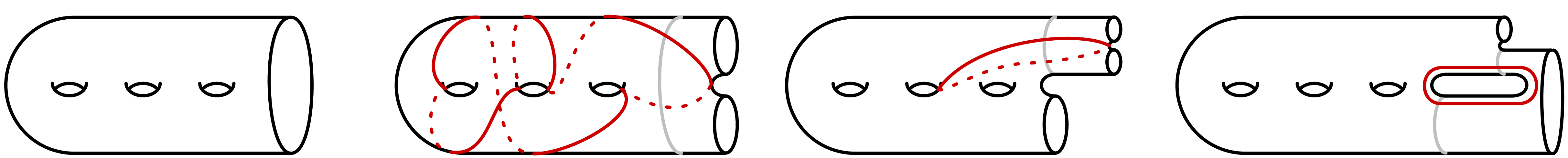
Simple generating sets

These come in a vast array of possibilities:

Start with the E_6 *configuration*



Now perform *any sequence* of “stabilizations”



The result generates the associated framed mapping class group!

Which one? The one uniquely specified by the condition that each distinguished curve has “zero holonomy” for the framing.

Linear systems on toric surfaces

Setup: X a smooth *toric surface* (e.g. $\mathbb{CP}^2, \mathbb{CP}^1 \times \mathbb{CP}^1$)

L an ample line bundle (e.g. $\mathcal{O}(d)$ on $\mathbb{CP}^2$)

$\mathscr{X}$ the family of *smooth* sections of L (e.g. smooth deg.-d plane curves)

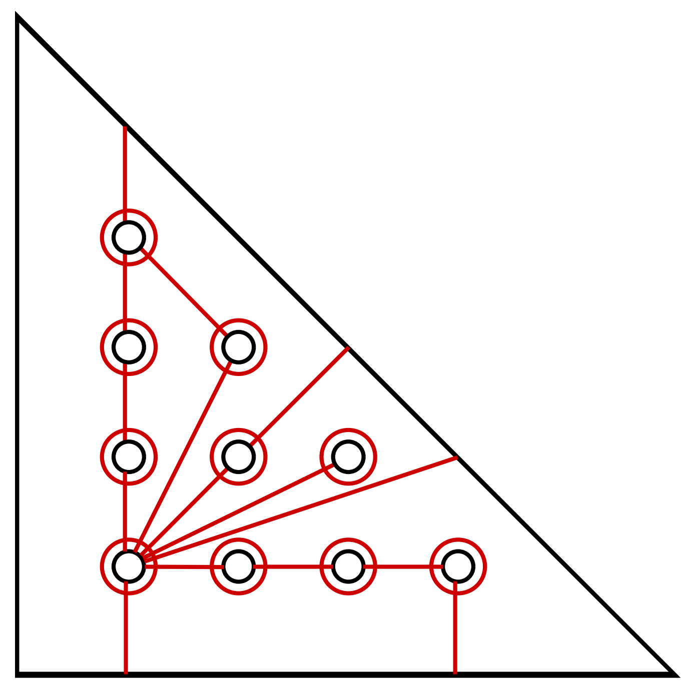
Theorem (S.): If the generic section is not hyperelliptic, then the monodromy of this family is $\mathrm{Mod}(\Sigma)[\phi]$ for ϕ the r-spin structure associated to the maximal root of the adjoint line bundle $K_X \otimes L$.

(e.g. $r = d - 3$ for deg.-d plane curves)

A word on the proof

Part 1: Build a model "reference fiber", and construct a sufficiently large supply of Dehn twists in the monodromy.

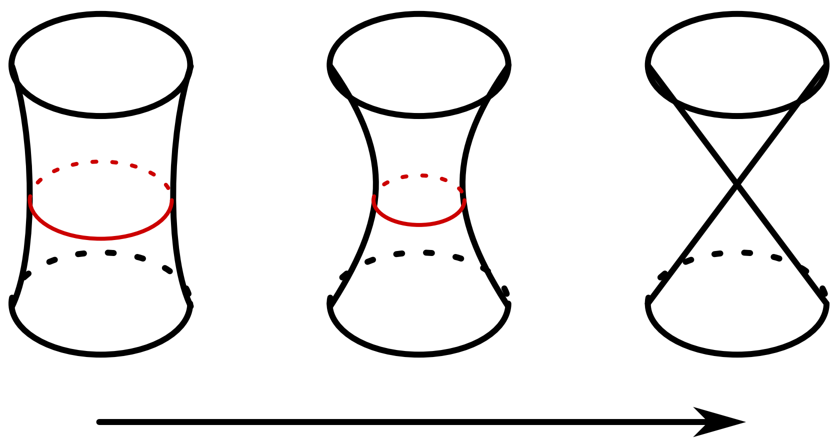
Crétois-Lang: Build *polygonal* models for reference fibers in toric surfaces, using methods of tropical geometry



Part 2: Invoke the Main Theorem to conclude this collection generates the r-spin mapping class group.

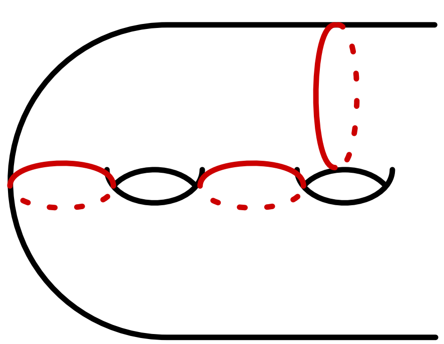
Corollaries (I)

Question (Donaldson, '00): Which curves can be the *vanishing cycles* associated to nodal degenerations?



Answer (S.): A curve $c \subset \Sigma$ is a vanishing cycle if and only if it satisfies $\phi(c) = 0$

For instance, this configuration is prohibited:



Proof: Associate VC's to Dehn twists in the monodromy; classify the latter

Translation surfaces

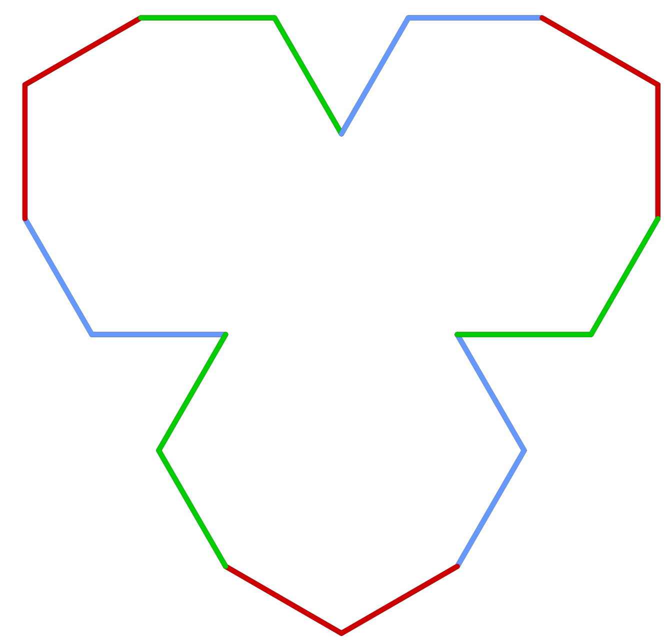
Algebraic geometry

(X,ω): X Riemann surface,
ω a holomorphic 1-form

$$\left(X^3 + Y^4 = 1,\ \frac{dX}{Y^3}\right)$$

Geometric geometry

Surface with atlas of charts to $\mathbb{C}$,
transitions $z \mapsto z + c$ (translations)



dz

$$x \mapsto \int_{x_0}^{x} \omega$$

These are the same thing!

Strata

A *stratum* parameterizes all translation surfaces of the same “geometric type”

Every differential has 2g-2 zeroes (with multiplicity).
Geometry: cone points of flat metric

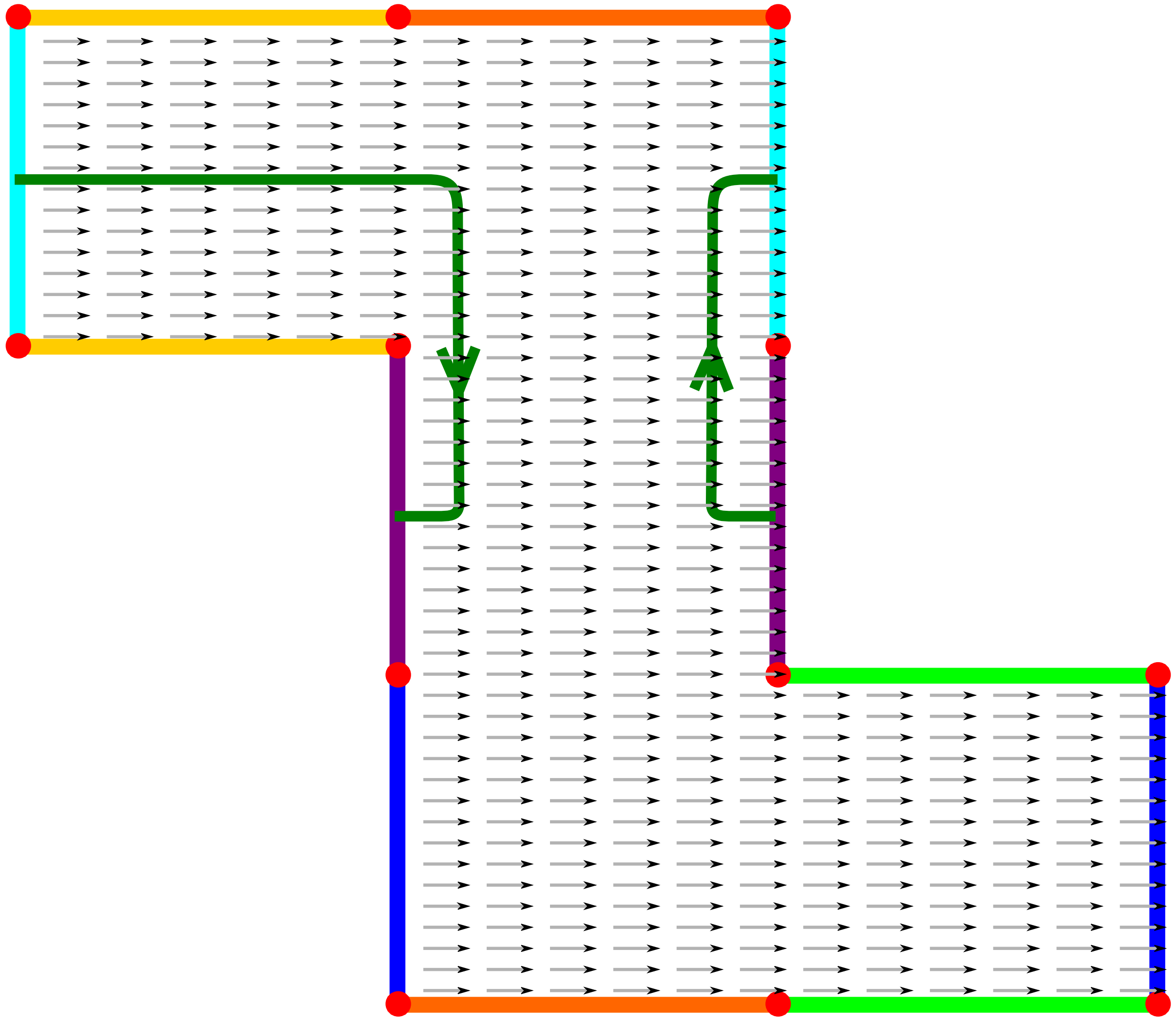
$\kappa = \{\kappa_1, \ldots, \kappa_n\}$: partition of 2g-2

Stratum $\mathscr{H}\Omega(\kappa)$: all translation surfaces with cone angle set κ

Period coordinates: each $\mathscr{H}\Omega(\kappa)$ is a complex orbifold of dimension 2g+n-1

Translation surfaces are framed

As remarked above, $C \backslash Z(\omega)$ carries the non-vanishing 1-form ω and hence is framed.



Monodromy of strata

Theorem (Calderon - S.):

Fix g ≥ 5, κ partition of 2g-2, and $\mathscr{H} \subseteq \mathscr{H}\Omega(\kappa)$ "non-hyperelliptic"*. Then

$$\rho_{\mathscr{H}} : \pi_1(\mathscr{H}) \to \mathrm{Mod}(\Sigma)[\phi]$$

is surjective. Here, ϕ is the framing associated to the horizontal vector field.

Corollaries:

- Classification of components over Teichmüller space
- Action on relative periods
- Which curves can be cylinders? Which arcs can be saddles?
- Starting point for study of $\pi_1(\mathscr{H})$

*: this is the generic case (hyperelliptic is classically understood)