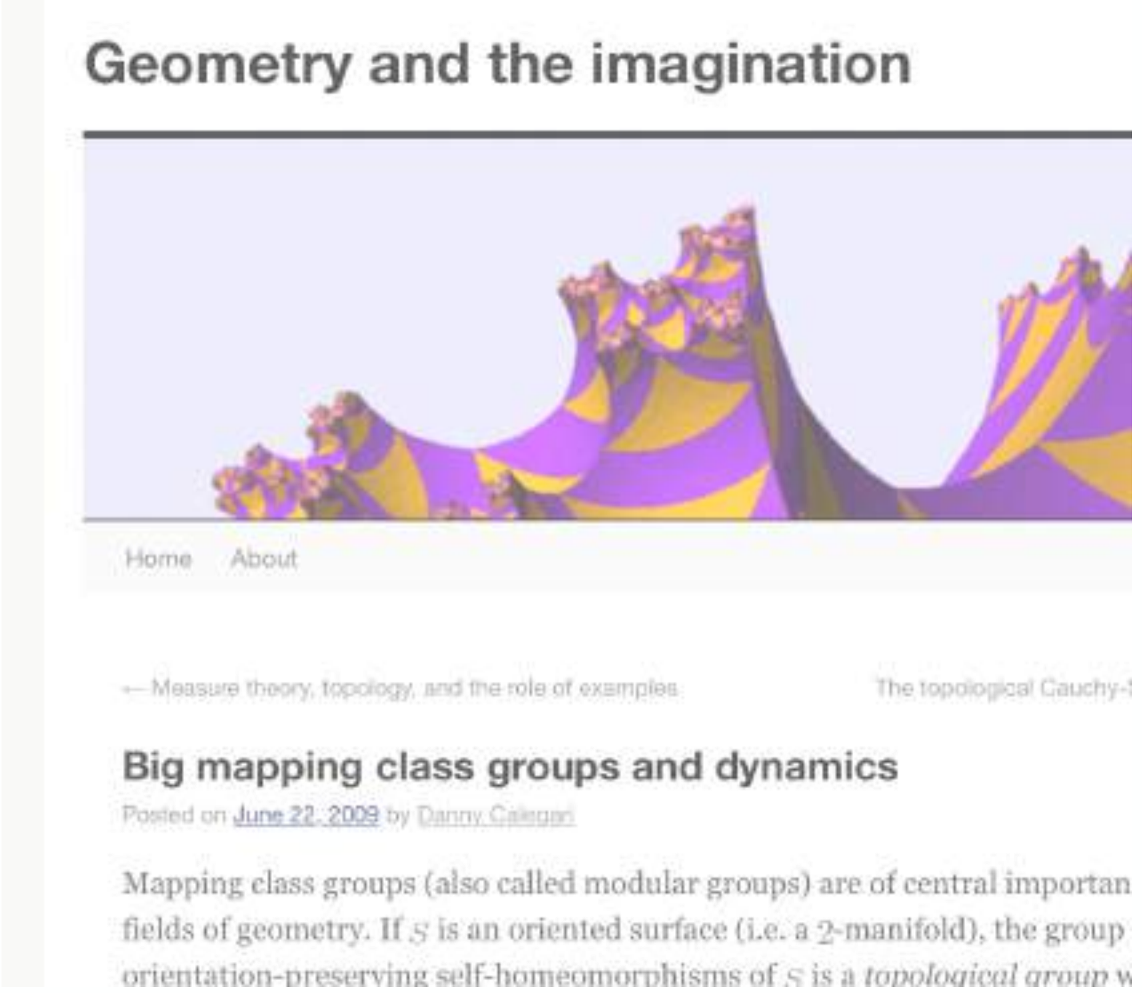
Large scale geometry of big mapping class groups

Kathryn Mann
joint work with Kasra Rafi

(ultra brief) History


Geometry and the imagination

Home About

← Measure theory, topology, and the role of examples

The topological Cauchy-

Big mapping class groups and dynamics

Posted on June 22, 2009 by Danny Calegari

Mapping class groups (also called modular groups) are of central importan
fields of geometry. If S is an oriented surface (i.e. a 2-manifold), the group
orientation-preserving self-homeomorphisms of S is a topological group w

cantor set

- **2009** Calegari: $\mathrm{MCG}(S^2 - C)$ has <u>bounded commutator length</u>

  asks: same for $\mathrm{MCG}(\mathbb{R}^2 - C)$?

- **2016** Bavard: no, $\mathrm{MCG}(\mathbb{R}^2 - C)$ acts on hyperbolic graph...

  ...can build nontrivial quasi-morphism

- **Since then:** Many attempts to answer
  - Which MCG's act on hyperbolic spaces?
  - Which admit unbounded *length functions*?

$$\boxed{\begin{aligned} &F : G \to \mathbb{R} \\ &\cdot\ F(ab) \le F(a) + F(b) \\ &\cdot\ F(id) = 0 \\ &\cdot\ F(a^{-1}) = F(a) \end{aligned}}$$

Unifying question

Which MCG’s have some intrinsic, nontrivial geometry?

Unifying question

Which MCG's have some intrinsic, nontrivial geometry?

Theorem [M— Rafi]: *an answer to this question*

Challenges: 1 hard to say anything about all surfaces

2 what does intrinsic geometry even mean?

3 (actual details of actual proof)

1 Classification of surfaces [I. Richards]

Genus $(\mathbb{N} \cup \{\infty\})$

Space of ends ($\cong$ closed subset of cantor set)

Space of ends accumulated by genus (further closed subset)

① Classification of surfaces [I. Richards]

- Genus ($\mathbb{N} \cup \{\infty\}$)
- Space of ends ($\cong$ closed subset of cantor set)
- Space of ends accumulated by genus (further closed subset)

* Classifying such objects is hard (in the complexity theory sense)

[Ketonen '78] – Gives a description of closed subsets of Cantor up to homeomorphism

[Camerlo - Gao '00] – Show that this is optimal / "simplest possible"

2 Intrinsic geometry: boundedness

(easier to say what "trivial geometry" is than what "geometry" is.)

2 # Intrinsic geometry: boundedness

(easier to say what "trivial geometry" is than what "geometry" is.)

Def: G is *coarsely bounded* if every continuous length function on G is bounded

topological group

(equivalently, every continuous action on a metric space has bounded orbits)

2 Intrinsic geometry: boundedness

(easier to say what "trivial geometry" is than what "geometry" is.)

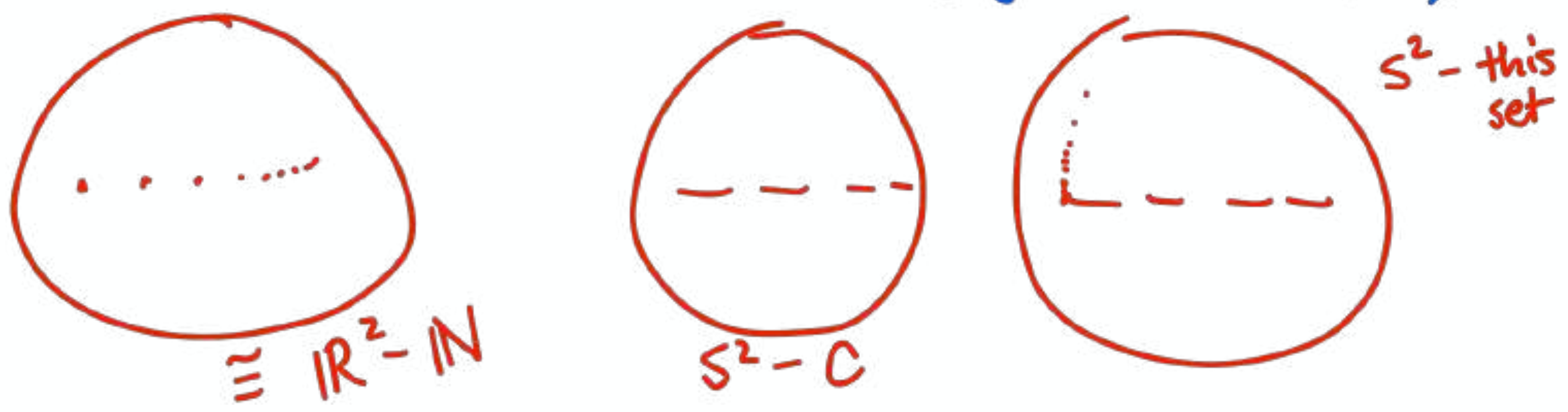
Def: G is *coarsely bounded* if every continuous length function on G is bounded

topological group

(equivalently, every continuous action on a metric space has bounded orbits)

Theorem: If Σ is *self-similar,* or *telescoping,* then $\mathrm{MCG}(\Sigma)$ is coarsely bounded

- genus 0 or ∞
- If you partition Ends(Σ) into 2 pieces, one contains a copy of Ends (Σ)



example:

Ends = Cantor set $\cup$ Cantor set accumulat by genus

Theorem: For surfaces with *tame end space* this is an iff

Proof sketch: $\mathrm{MCG}(S^2 - C)$ **is coarsely bounded**

1. ℓ continuous $\Rightarrow$ $\exists$ nbhd V of id. where $\ell(V) \subset [0, 1]$

2. Subadditivity $\Rightarrow$ suffices to show $\exists\, f_1, \dots f_n$ s.t. any mapping class can be written as word of length ≤ 10 in f_i's and V.

3. (Sketch)

2'

Intrinsic geometry: non-boundedness

Def: A subsurface $S \subset \Sigma$ is *displaceable* if $f(S) \cap S = \varnothing$ for some f

Theorem: If Σ has a non-displaceable finite-type subsurface, then MCG(Σ) is not coarsely bounded

Examples:

- Σ finite genus (put all the genus in S)
- 
  $= \Sigma$
- Ends = 

Idea of proof:

- WLOG S has unbounded curve graph
- Take $\mu \subset \mathcal{C}(S)$ filling
- Use subsurface projection of $\phi(\mu)$ to S to define a length function
- If ϕ restricts to p.A. on S, easy to see unbounded on ϕ^n.

2'' Intrinsic geometry: general framework

Geometric group theory works for *locally compact, compactly generated* groups

Rosendal showed this can be extended to topological groups that are
locally coarsely bounded and generated by a *coarsely bounded set*

(neighborhood of id.)

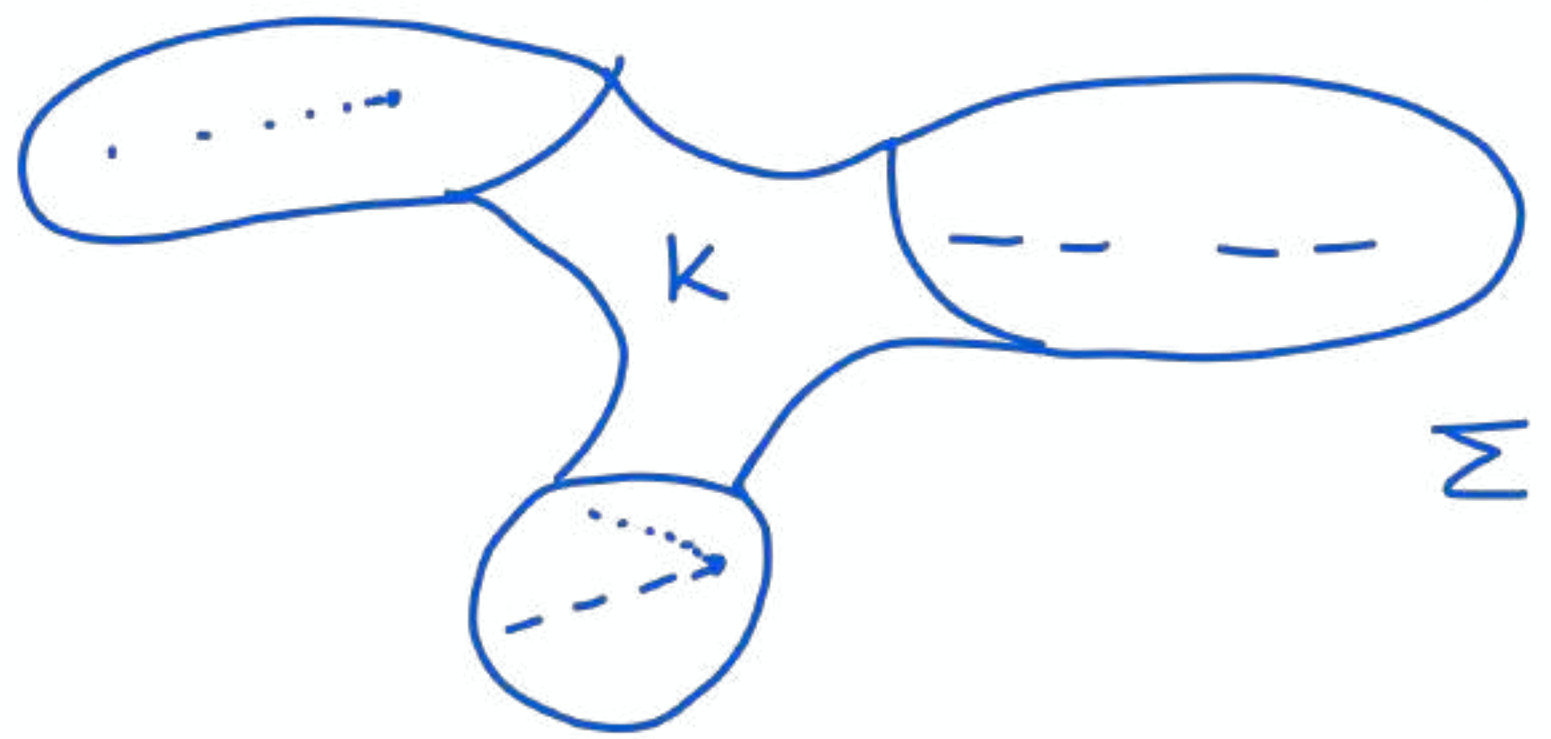
(analytic)

(the word metric for such a generating set gives a well-defined coarse geometric structure)

Q: which big mapping class groups fit this framework?

C.B. neighborhood of identity

Theorem: MCG(Σ) has a CB neighborhood of id. $\Leftrightarrow$ $\exists$ finite type $K \subset \Sigma$ partitioning Ends into self-similar sets *



In this case, {mapping classes that are trivial on K} is a CB nbhd of id

C.B. neighborhood of identity

Theorem: $MCG(\Sigma)$ has a CB neighborhood of id. $\Leftrightarrow$ $\exists$ finite type $K \subset \Sigma$ partitioning Ends s.t.

- Ends $= \bigsqcup A_i \sqcup \bigsqcup P_j$

  A_i: self similar sets

  P_j: Each $\cong$ some piece of some A_i, with $A_i \cup P_j \cong A_i$

- "Most complicated" ends appear in the A_i's and for any such end $\varepsilon \in A_i$, and nbhd V of ε, $\exists f$ s.t. $f(V) \supset A_i$ } "expansivity" condition

In this case, {mapping classes that are trivial on K} is a CB nbhd of id

C.B. neighborhood of identity

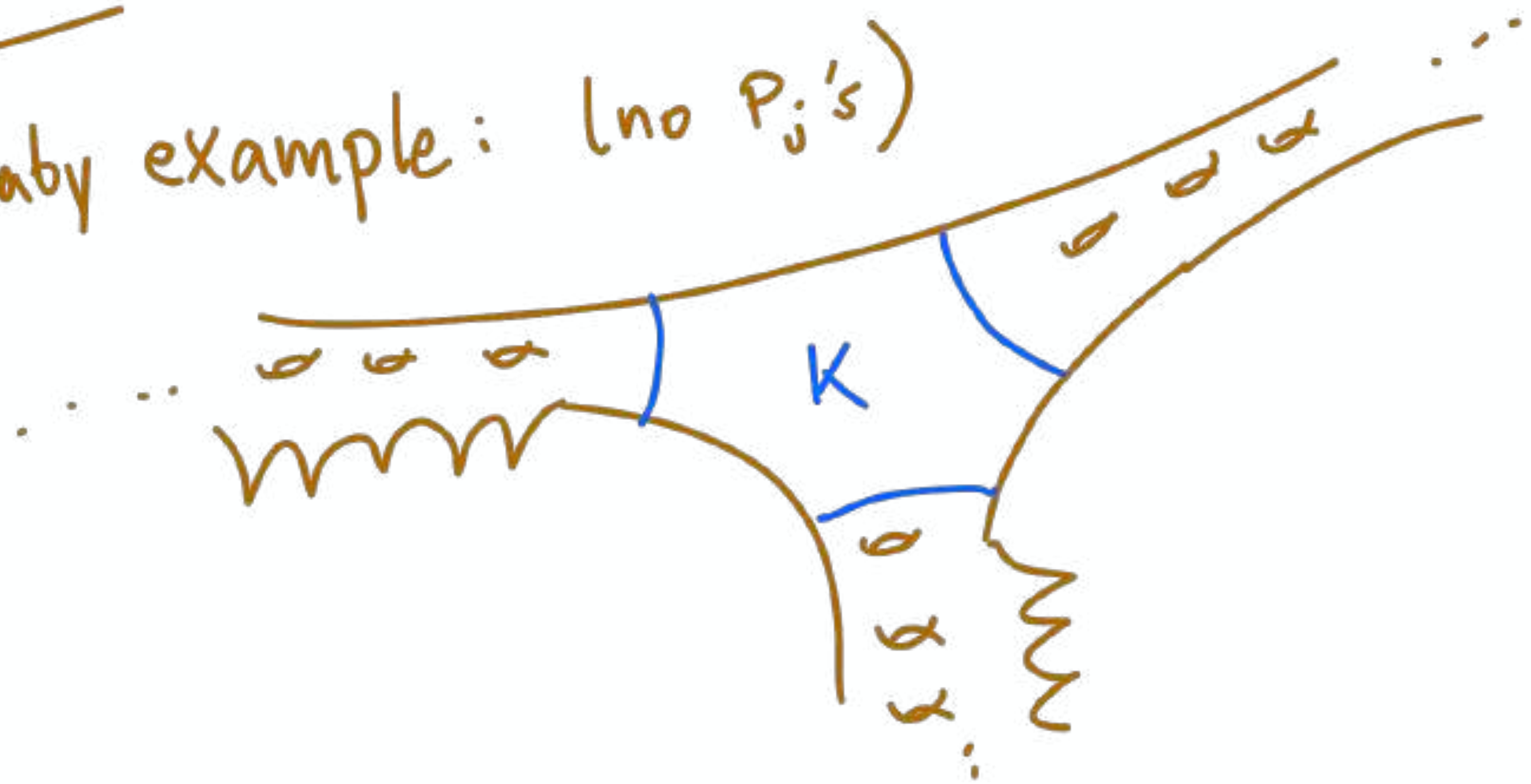
Theorem: $MCG(\Sigma)$ has a CB neighborhood of id. $\Leftrightarrow$ $\exists$ finite type $K \subset \Sigma$ partitioning Ends s.t.

- $\text{Ends} = \underbrace{\bigsqcup A_i}_{\text{self similar sets}} \sqcup \underbrace{\bigsqcup P_j}_{\text{Each} \cong \text{some piece of some } A_i,\ \text{with } A_i \cup P_j \cong A_i}$

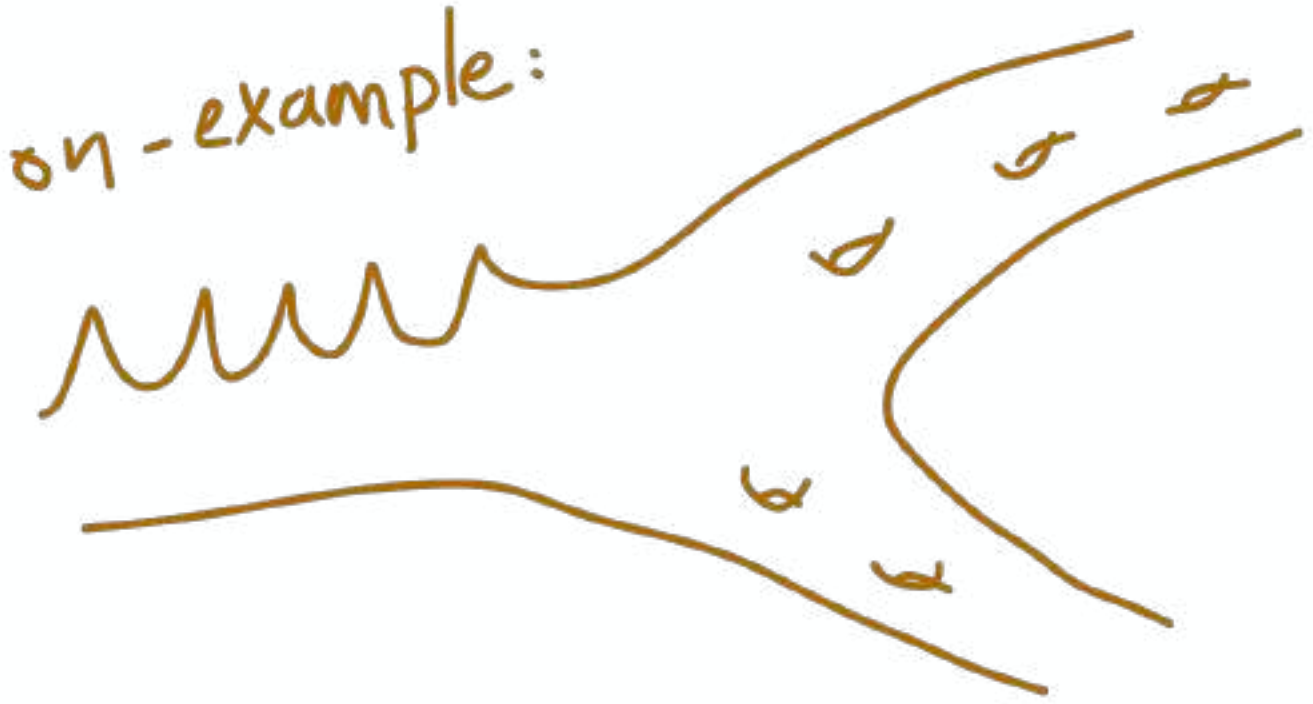
- "Most complicated" ends appear in the A_i's and for any such end $\varepsilon \in A_i$, and nbhd V of ε, $\exists f$ s.t. $f(V) \supset A_i$

In this case, {mapping classes that are trivial on K} is a CB nbhd of id

Baby example: (no P_j's)



Non-example:



C.B. generating set

- Not every locally C.B. group has a C.B. generating set

- Not every C.B.-generated group is locally C.B.

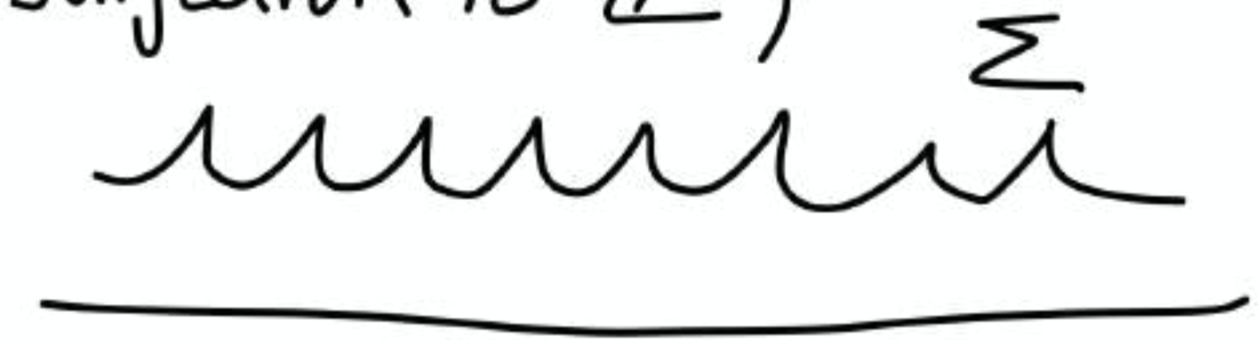
But for Polish groups, CB-gen $\Rightarrow$ locally CB

Two things that prevent C.B. generation

1. ***Infinite rank***

There are surfaces where (finite index subgroup of) $MCG(\Sigma) \twoheadrightarrow \mathbb{Z}^n$ for all n.

Ex: (Surjection to $\mathbb{Z}$)



Ends = Countable set with 2 accumulation points.

Two things that prevent C.B. generation

2. ***Limit type*** **behavior**

Countable end spaces (no genus) $\longleftrightarrow$ ordinals with order topology $\omega^\alpha \cdot n$

$\omega^\alpha + \omega^\alpha$ end space, α <u>limit ordinal</u> cannot be CB generated.

<u>THEOREM:</u> for "tame" surfaces, these are the only obstructions.

Classification theorem

<u>Thm</u> : Among tame surfaces,

$MCG(\Sigma)$ has a well-defined Q.I. type $\Longleftrightarrow$ • Either it is globally C.B. (trivial geometry)

or

• Locally CB + "Finite Rank"

&

not "Limit type".

Key tool: complexity of an end

Def: for $x, y \in \mathrm{Ends}$,

$x \preceq y$ if $\forall$ nbhd U of y, $\exists$ subset $V \subset U$ homeomorphic to a nbhd of x.

$x \preceq y$ and $y \preceq x$ $\not\Rightarrow$ $\exists$ nbhd of y homeomorphic to nbhd of x

"TAME" is the requirement that this ↑ doesn't happen for maximal ends (wrt. $\preceq$) & their immediate predecessors.

(all concrete examples we have seen in other papers are tame, but we can painfully construct some non-tame ones!)

Complexity: examples and properties

ω

ω^2

- $\preccurlyeq$ "agrees with usual order on ordinals"
- maximal ends
- easy now to construct nondisplaceable subsurfaces.