Ralf Meyer, Classifying C^*-algebras up to KK-equivalence
with extra structure equivariant

15.1.2025 Deep analysis: under extra assumptions, KK^G-equivalence implies equivariant Morita equivalence

Sample result: Consider two strongly purely infinite, nuclear, separable, $\mathcal{O}_\infty$-absorbing C^*-algebras A, B with $\mathrm{Prim}(A) \cong \mathrm{Prim}(B) = X$ ($\mathrm{Ideals}(A) \cong \mathrm{Ideals}(B)$)

Any equivalence in $KK(X; A, B)$ lifts to a Morita equivalence.

<u>Example</u> An extension of simple C^*-algebras $I \rightarrowtail A \twoheadrightarrow B$,
I, B simple $\mathrm{Prim}(A) = \{\circ \ \circ\}$

$KK(X)$-equivalence $\iff$ isomorphism between the K-theory long exact sequences

Can get this from the homological algebra machinery using the generators
$\mathbb{C} = \mathbb{C} \to 0$ $0 \to \mathbb{C} = \mathbb{C}$ $S\mathbb{C} \rightarrowtail C\mathbb{C} \twoheadrightarrow \mathbb{C}$ to produce a UCT

This simple Ansatz works well if you have an increasing chain of finitely many ideals. $0 = I_0 \triangleleft I_1 \triangleleft I_2 \triangleleft \dots \triangleleft I_n = A$ with simple subquotients. $K_*(I_k / I_\ell)$ for $\ell < k$

How to go beyond these cases?

There are two rather effective methods
- allow length-2 projective resolutions (Bentmann–M)
  add a class in Ext^2 to the data
- use KK-valued invariants

<u>Example</u> $KK^{\mathbb{Z}}$ or $\mathbb{Z}$-actions on C^*-algebras
$KK^{\mathbb{Z}} \cong KK^{\mathbb{T}}$, so can also treat $KK^{\mathbb{T}}$.

The bootstrap class in $KK^{\mathbb{Z}}$ is generated by $C_0(\mathbb{Z})$ with $\mathbb{Z}$ acting by translation. In $KK^{\mathbb{T}}$ this becomes $C_0(\mathbb{Z}) \rtimes \mathbb{Z} \cong \mathbb{K}(\ell^2\mathbb{Z}) \sim \mathbb{C}$
$KK^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) = \mathrm{Rep}(\mathbb{T}) = \mathbb{Z}[x, x^{-1}]$
universal Abelian approximation $KK^{\mathbb{Z}} \xrightarrow{K_*(-)} \mathrm{Mod}(\mathbb{Z}[x, x^{-1}])^c_{\mathbb{Z}/2}$
has length-2-resolutions, but usually not length-1

Alternative: $KK^{\mathbb{Z}} \longrightarrow KK[\mathbb{Z}] \ni (A, \alpha)$ with $\alpha \in KK_0(A, A)^\times$
$\cap$
Abelian category $[\mathbb{Z}]$ length-1-resolutions

<u>Theorem</u> $\alpha \in \mathrm{Aut}\, A$, $\beta \in \mathrm{Aut}\, B$: (A, α) and (B, β) are $KK^{\mathbb{Z}}$-equival.
$\iff \exists\, \gamma \in KK_0(A, B)^\times$: $\beta \circ \gamma = \gamma \circ \alpha$ in $KK_0(A, B)^\times$

Why can classification with the obstruction class work for length-2 projective resolutions?
There is a spectral sequence computing $KK(A,B)$ ($A \in$ Bootstrap) involves $\mathrm{Ext}^p_{\mathcal{C}}(F(A), F(B))$ on the second page
by assumption, only $p=0,1,2$ occur, so the only possible boundary map goes from Ext^0 to Ext^2
$\delta_A \in \mathrm{Ext}^2_{\mathcal{C}}(F(A), F(A)[1])$ An isomorphism $F(A) \cong F(B)$ exists that goes to 0 in $\mathrm{Ext}^2(\cdot)$ if and only if there is an isomorphism that maps δ_A to δ_B.
Both approaches to get classification beyond the UCT work for
- $\mathbb{Z}$-actions, $\mathbb{T}$-actions
- graph C^*-algebras with finitely many ideals
- C^*-algebras with $\mathrm{Prim}(A) \cong X$ finite unique path space

How to classify actions of $\mathbb{Z}/p$ for a prime p? Köhler, M. Köhler found a UCT for this case using a certain generator for the bootstrap class.
$\mathbb{Z}/p$-actions in the equivariant bootstrap class / $KK^{\mathbb{Z}/p}$-equivalence
$\cong$ countable exact modules over $\mathbb{Z}/2$-graded ring R / isomorphism

How about actions on Cuntz algebras?
The result is simple if $p \in KK^G(A,A)$ is invertible. Then the invariant consists of three $\mathbb{Z}/2$-graded modules, countable, one is over $\mathbb{Z}[1/p]$, two over $\mathbb{Z}[\vartheta_p, 1/p]$, ϑ_p primitive pth root of unity.
$\mathbb{Z}[\mathbb{Z}/p][1/p] = KK^{\mathbb{Z}/p}(\mathbb{C},\mathbb{C})[1/p] \cong \mathbb{Z}[1/p] \oplus \mathbb{Z}[\vartheta_p, 1/p]$
Köhler's generators are $\mathbb{C}$, $C(\mathbb{Z}/p)$, $\mathrm{cone}(\mathbb{C} \hookrightarrow C(\mathbb{Z}/p))$
$R = KK^{\mathbb{Z}/p}$-end of $\mathbb{C} \oplus C(\mathbb{Z}/p) \oplus \mathrm{cone}(\mathbb{C} \hookrightarrow C(\mathbb{Z}/p)) = \mathcal{D}$
$C(\mathbb{Z}/p) \cong \mathbb{K}(\ell^2 \mathbb{Z}/p) \sim \mathbb{C}$ has cone $\Sigma\, \mathrm{cone}(\mathbb{C} \hookrightarrow C(\mathbb{Z}/p))$

$X \rightleftarrows Y \rightleftarrows Z \rightleftarrows \Sigma X$ $\qquad$ $0 \to Y = Y \to 0$

$\mathbb{Z}/p^n$ $\qquad$ Y p-divisible