Tensor triangular geometry for equivariant KK-theory

by Ivo Dell'Ambrogio, LML, Univ. Artois 15/01/25
j. work with Rubén Martos (arXiv:2412.2109)

Starting point:

- [Meyer-Nest 2006]: G a 2nd countable loc.cpt group
  $\rightsquigarrow$ the G-equivariant Kasparov category KK^G is a tensor triangulated category (tt-category).

Namely:

1) It is a symmetric monoidal category:
   objects = separable G-C^*-algebras
   $\mathrm{Hom}(A,B) = KK_0(A,B)$
   composition and $\otimes$ of maps: Kasparov product
   $A \otimes B = A \otimes_{\min} B$ (or $\otimes_{\max}$ ...) with diag. G-action

2) Triangulated:
   $A \to B \to C \to \Sigma A$ given by semi-split extensions or Puppe sequences

   (de)suspension: $\Sigma A = C_0(\mathbb{R}, A) = C_0(\mathbb{R}) \otimes A$

   $\rightsquigarrow$ Bott periodicity: $\Sigma \circ \Sigma \simeq \Sigma$

3) Some mild compatibility:
   $- \otimes - : KK^G \times KK^G \longrightarrow KK^G$
   is exact (preserves Δs) in both variables

This is the structure we will work with!

- tt-categories are a "light" axiomatization of (the homotopy category of) stable symm. mon. ∞-categories

- There is a powerful geometric theory of tt-cat's: tensor-triangular geometry (tt-geometry)

  Main tool [Balmer 2005]:

  - If $\mathcal{K}$ is an ess. small tt-category, its spectrum $\mathrm{Spc}(\mathcal{K})$ is a topological space.
  - Each object $A \in \mathcal{K}$ has a support $\mathrm{Supp}(A) \subseteq \mathrm{Spc}(\mathcal{K})$.
  - $A \mapsto \mathrm{Supp}(A)$ is compatible with the algebraic op's.

Theorem: suppose $\mathcal{K}$ is rigid (each $A \in \mathcal{K}$ has a $\otimes$-dual). There is an inclusion preserving bijection:

$$\left\{ \begin{array}{c} \text{thick } \otimes\text{-ideal} \\ \text{subcategories } \mathcal{C} \subseteq \mathcal{K} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{Thomason subsets} \\ S \subseteq \mathrm{Spc}(\mathcal{K}) \end{array} \right\}$$

given by: $\mathcal{C} \longmapsto \mathrm{Supp}(\mathcal{C}) := \bigcup_{A \in \mathcal{C}} \mathrm{Supp}(A)$

$\mathcal{C}$ thick : Δed subcat closed under retracts

S Thomason : can write $S = \bigcup_i Z_i$, Z_i^c quasi-cpt. open $\forall i$

- Get a rough classification of the objects of $\mathcal{K}$:

  $\mathrm{Supp}(A) = \mathrm{Supp}(B) \iff \mathrm{Thick}_\otimes(A) = \mathrm{Thick}_\otimes(B)$

  $\iff$ can build A and B from each other using the tt-operations (cones, $-\otimes C$ ...)

- To apply this: must compute $\mathrm{Spc}(\mathcal{K})$ in examples!

Some well-known examples:

(1) X a quasi-cpt & quasi-sep scheme, $\mathcal{K} = D^{perf}(X)$

By Thomason: $\mathrm{Spc}(D^{perf}(X)) \cong X$

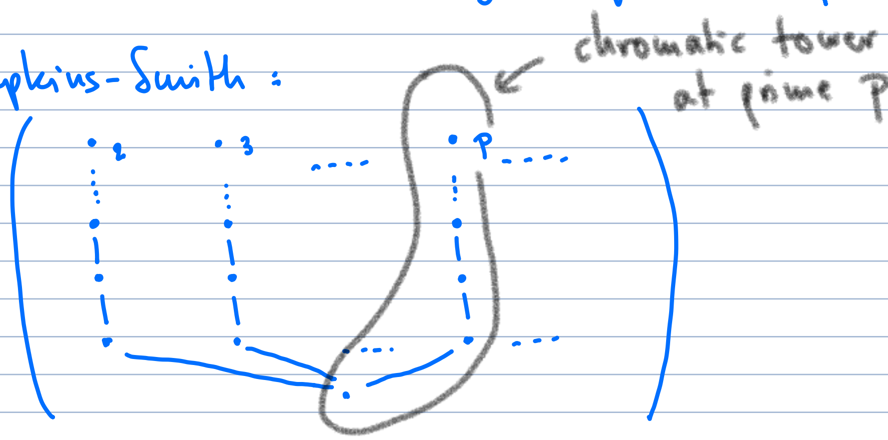
Affine case $X = \mathrm{Spec}_{Zar}(R) \rightsquigarrow \mathrm{Spc}(K^b(\mathrm{proj}\, R)) \cong \mathrm{Spec}_{Zar}(R)$.

(2) G finite group, k field, $\mathcal{K} = \mathrm{stab}(kG) = \frac{\mathrm{mod}\, kG}{\mathrm{proj}\, kG}$ (add)

By Benson-Carlson-Rickhard: $\mathrm{Spc}(\mathcal{K}) \cong \mathrm{Proj}(H^*(G;k))$.

(3) $SH^c = \mathrm{Ho}(Sp_\omega)$ homotopy category of finite spectra

By Devinatz-Hopkins-Smith:

$\mathrm{Spc}(SH^c) =$ 

- So what about $\mathcal{K} = KK^G$?

No idea, even for $G = 1$ trivial ☹

Too bad, because enough knowledge of $\mathrm{Spc}(KK^G)$ would decide the "very strong" Baum-Connes conjecture $(\gamma_G = 1_G)$ ...

- Many problems:
  - KK^G not rigid, in fact it can have $\otimes$-nilpotent objects.
  - Too many objects; no sensible set of generators!
  - It has infinite coproducts $\coprod_\infty$, so it makes more sense to classify localizing $\otimes$-ideal subcategories.

  localizing: also closed under $\coprod_\infty$'s

- Therefore: let's try to classify localizing $\otimes$-ideals in a "reasonably generated" sub-tt-category of KK^G

Also: suppose G finite (some results also for G cpt...)

- An interesting but reasonable choice is the G-equivariant Bootstrap category [D.-Emerson-Meyer '14]:

$$\mathrm{Boot}^G \overset{\text{def.}}{=} \{ A : A \text{ is } KK^G\text{-equiv. to a Type I sep. } C^*\text{-alg.} \} \underset{\text{full}}{\subseteq} KK^G$$

$$= \mathrm{Loc}\left(\{ A : A = \mathrm{Ind}_H^G (H \overset{\alpha}{\curvearrowright} M_n(\mathbb{C})) \} \right)$$

$$= \mathrm{Loc}\left(\{ C(G/H) : H \leq G \text{ is a cyclic subgroup} \} \right)$$

by Arano-Kubota (2018) + Meyer-Nadareishvili (2024)

- Boot^G is a nicer tt-category with countable $\coprod_\infty$.

For instance:

$$\mathrm{Boot}^G_c = \mathrm{Boot}^G_d \text{ is a rigid ess. small tt-cat.}$$

the $\otimes$-dualizable objects

compact objects $A \in \mathrm{Boot}^G$:
$\mathrm{Hom}(A,-)$ preserves ∞ coproducts

Moreover:

$$A \text{ cpt-rigid} \Rightarrow K_*^H(A) \text{ is a fin. gen. } R(G)\text{-module } \forall \text{ subgroup } H \leq G.$$

- For some G, we can classify:

① the thick $\otimes$-ideals of $\mathrm{Boot}^G_c \overset{\text{Balmer}}{\longleftrightarrow}$ compute $\mathrm{Spc}(\mathrm{Boot}^G_c)$

② the localizing $\otimes$-ideals of Boot^G !

- **Def**: a finite group G is of prime order elements if every non-trivial $g \in G$ has prime order
  ($\iff$ the non-triv. cyclic $H \leq G$ have prime order).

- **Examples**: $\mathbb{Z}/p\mathbb{Z}$ p prime, $(\mathbb{Z}/p\mathbb{Z})^n$, S_3, A_5, ...

- **Theorem A** [D.-Martos 2024]
  If G is finite of prime order elements, $\exists$ canonical homeo
  $$\mathrm{Spc}(\mathrm{Boot}_c^G) \cong \mathrm{Spec}_{\mathrm{Zar}}(R(G)).$$

- **Theorem B** [D. Martos 2024]
  For every finite G, the Balmer support theory for Boot_c^G admits an extension (with nice prop's) to arbitrary $A \in \mathrm{Boot}^G$:
  $$\mathrm{Supp} : \mathrm{Obj}(\mathrm{Boot}^G) \longrightarrow \{\text{subsets of } \mathrm{Spc}(\mathrm{Boot}_c^G)\}.$$
  If G has prime order elements, it induces a bijection:
  $$\left\{ \begin{array}{c} \text{localizing } \otimes\text{-ideals} \\ \text{of } \mathrm{Boot}^G \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \text{arbitrary subsets} \\ \text{of } \mathrm{Spec}_{\mathrm{Zar}}(R(G)) \end{array} \right\}.$$

- We fully expect both Theorems to hold for all finite G.
- In fact, Arano-Kubota (+ "stratification theory") lets us reduce both simultaneously to the case of G cyclic!
- If $G \cong \mathbb{Z}/p\mathbb{Z}$, essentially proved by D.-Meyer (2021)

The case of G cyclic of any order n, and therefore of general finite G, remains unproven...

Ideas for the proofs

For Thm A

- The homeo is via a natural continuous map

$$\rho_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathrm{Spec}_{\mathrm{Zar}}(\mathrm{End}_{\mathcal{K}}(\mathbb{1}))$$

$\otimes$-unit of $\mathcal{K}$

which exists $\forall$ ess. small tt-cat. $\mathcal{K}$ (and in general is neither inj. nor surj.).

- Using general tt-geometry, Arano–Kubota, and the structure of $\mathrm{Spec}_{\mathrm{Zar}}(R(G))$ [Segal 1968], $\rightsquigarrow$ reduce to $G = \mathbb{Z}/p\mathbb{Z}$, p prime.
Then use Köhler's UCT for $\mathrm{Boot}^{\mathbb{Z}/p\mathbb{Z}}$, see [D.-Meyer 2021].

For Thm. B

- Use stratification theory (Hovey–Palmieri–Strickland, Neeman, Benson–Iyengar–Krause, and especially Barthel–Heard–Sanders 2023)

- Setting for this:

Suppose $\mathcal{K} := \mathcal{T}_c = \mathcal{T}_d \subseteq \mathcal{T}$ ⟸ "a rigidly-compactly generated tt-cat"

a nice rigid ess. small tt-category

e.g.: $\mathcal{T} = \mathrm{Ho}(\mathcal{C})$, $\mathcal{C}$ a presentably sym. mon. stable ∞-cat, gen. by a set of rigid-cpts

Suppose also $\mathrm{Spc}(\mathcal{K})$ is a (weakly) noetherian space, e.g. $\cong \mathrm{Spec}_{Zar}(R)$ of a noetherian commutative ring like $R(G)$, G finite.

Then:

- [Balmer-Favi 2011]

  $\exists$ a nice support theory for all $A \in \mathcal{T}$:
  $$\mathrm{Supp}(-) : \mathrm{Obj}(\mathcal{T}) \to \mathrm{Subsets}(\mathrm{Spc}(\mathcal{K})) .$$

  $\rightsquigarrow$ It induces a surjective map

  $$\left\{ \begin{matrix} \text{localizing } \otimes\text{-ideals} \\ \text{of } \mathcal{T} \end{matrix} \right\} \to \left\{ \begin{matrix} \text{subsets of} \\ \mathrm{Spc}(\mathcal{K}) \end{matrix} \right\} .$$
  $$\mathcal{L} \mapsto \bigcup_{A \in \mathcal{L}} \mathrm{Supp}(A)$$

  $\mathcal{T}$ is "stratified"

- Thm (B-H-S 2023)
  - The map is also injective, hence bijective, provided that $\forall\ \mathfrak{p} \in \mathrm{Spc}(\mathcal{K})$, the loc. subcat. $\mathcal{L}_{\mathfrak{p}}$ "supported at $\mathfrak{p}$" is minimal.
  - This minimality can be checked locally in $\mathrm{Spc}(\mathcal{K})$, i.e. one $\mathfrak{p}$ at a time, via various kind of procedures ...

- For $\mathrm{Boot}^{\mathbb{Z}/p\mathbb{Z}}$, can do "by hand" as $\mathrm{Spc}(\mathcal{K}) \cong \mathrm{Spec}(\mathbb{Z}[x]/_{\frac{x^p-1}{x-1}})$ is small ...

  Thm. A

- Again, Arano–Kubota lets us deduce from this the case of G of prime order elements

⚠ Problem: this stratification theory only applies for $\mathcal{T}$ with arbitrary small coproducts!

But Boot^G only has countable ones (and is rigidly-compactly generated in a weaker sense ...)

- Solution: Use ∞-categorical enhancements to add small coprods to Boot^G

As explained by Ulrich Bunke (±):

$\exists$ stable symm. mon. ∞-cat KK^G_∞ with

$$Ho(KK^G_\infty) \cong KK^G .$$

all small $\amalg$'s

$$\rightsquigarrow \quad \mathrm{Boot}^G \hookrightarrow Ho(\mathrm{Ind}_{\omega_1}(KK^G_\infty)) \supset \mathrm{Loc}(\{C(G/H)\}) = \mathcal{T}$$

full tt-subcat., countable $\amalg$ are preserved

& $\mathrm{Boot}^G_c \xrightarrow{\simeq} \mathcal{T}_c$

apply stratification theory to this one!

- Finally, must check that if the big $\mathcal{T}$ is stratified, then Boot^G is also stratified in the "countable" sense.

(OK $\forall G$ finite)

QED