Twisted K-theory via homotopical algebra

Motivation: M cpt. smooth orientable mfd. $\partial M = \emptyset$

$\dim(M) = n$

$[M] \in H_n(M, \mathbb{Z})$ fundamental class

~~orient~~able : $[M] \in H_n(M, \mathcal{Z})$ ← local coeff. system

↑ homology w/ loc. coeff.

$$\mathcal{Z} = \bar{M} \times_{\mathbb{Z}/2} \mathbb{Z} \longrightarrow M$$

=

M spin^c-mfd. (K-orientable)

$[D_M] \in K_n(M)$ fundamental class

K-~~orient~~able : $[D_M] \in K^n(\Gamma(M, \mathbb{C}\ell(M)))$

↑ K-homology w/ loc. coeff.

<u>Def</u>. of twisted K-theory:

Let $\mathcal{A} \to X$ be a bundle of C^*-algebras over a cpt. Hausd. space X

$$K^n_{\mathcal{A}}(X) = K_n(\Gamma(X, \mathcal{A}))$$

•) note that $K^*_{\mathcal{A}}(X)$ is a module over $K^*(X)$.

Donovan-Karoubi '70 : $\mathcal{A} \to X$ bundles of matrix alg.

$$Br_u(X) \cong \mathrm{Tor}(H^3(X, \mathbb{Z}))$$

Rosenberg '88, Atiyah-Segal '05 : $\mathcal{A} \to X$ bundles of compact operators (or projective Hilbert space bundles)

$$[X, B\mathrm{Aut}(\mathbb{K})] \cong H^3(X, \mathbb{Z})$$

Thm (Freed-Hopkins-Teleman) : G cpt. simple, simply-conn. Lie group, $m \in H^3(G, \mathbb{Z}) \cong \mathbb{Z}$

$$K^{\dim G}_{\mathcal{K}(m), G}(G) \cong R_m(LG)$$

The homotopical picture

$$K : C^*Alg \longrightarrow \mathrm{Mod}(KU)$$

$K(\Gamma(X, \mathcal{A}))$ spectrum-valued K-theory

Idea : Given $\mathcal{A} \to X$ take $K(\mathcal{A}_x)$ fibrewise to get a "bundle" of KU-module spectra. Then take global sections.

Def. : local system :

$\mathcal{C}$ ∞-category

X obj. in $\mathcal{C}$

$BAut_e(X)$ full subcategory of $\mathcal{C}^{\simeq}$ on objects that are equiv. to X.

cat. of local systems on $W \in Spc$ with fibre X is $Fun(W, BAut_e(X))$.

Have a localisation functor $\ell : Top \longrightarrow Spc$ loc. at w. eq.

$Simp(X)$ simplex category of X

obj.: $\sigma : \Delta^n \longrightarrow X$

mor.: $\nu : [n] \longrightarrow [\tilde{n}]$ s.th.

$$\begin{tikzcd}
\Delta^n \arrow[rr, "|\nu|"] \arrow[dr, "\sigma"'] & & \Delta^{\tilde{n}} \arrow[dl, "\tilde{\sigma}"] \\
& X &
\end{tikzcd}$$

$$\ell(X) = \operatorname*{colim}_{Simp(X)} *$$

canonical functor $Simp(X) \longrightarrow \ell(X)$ is a Dwyer-Kan loc. at all morphisms in $Simp(X)$.

Crucial observation:

bundle $\mathcal{A} \longrightarrow X$ with fibre A gives rise to loc. system

1) functor $\phi_{\mathcal{A}} : Simp(X)^{op} \longrightarrow C^*Alg$

$$\phi_{\mathcal{A}}(\sigma : \Delta^n \to X) = \Gamma(\Delta^n, \sigma^*\mathcal{A})$$

$\phi_{\mathcal{A}}(\gamma)$ = corresp. restr. maps

2) Note that $\Gamma(\Delta^n, \sigma^*\mathcal{A}) \cong C(\Delta^n, A) \simeq A$

($\mathcal{A}$ is loc. trivial)

$$\begin{tikzcd}
\mathrm{Simp}(X)^{op} \arrow[r] \arrow[d] & C^*Alg \arrow[d] \\
\ell(X) \simeq \ell(X)^{op} \arrow[r] \arrow[dr, bend right, "\tilde{\phi}_{\mathcal{A}}"'] & C^*Alg_h \\
& BAut_{C^*Alg_h}(A) \arrow[u]
\end{tikzcd}$$

3) $\tilde{\phi}_{\mathcal{A}}$ induces $\begin{tikzcd} \ell(X) \arrow[r, "\tilde{\phi}_{\mathcal{A}}"] \arrow[rr, bend right, "K(\mathcal{A})"'] & BAut_{C^*Alg_h}(A) \arrow[r, "K"] & BAut_{Mod(KU)}(K(A)) \end{tikzcd}$

and we define

$$\Gamma(X, K(\mathcal{A})) = \lim_{\ell(X)} K(\mathcal{A}) \text{ in } Mod(KU)$$

Prop. (Bunke - Land - P.): If X is homotopically finitely dominated, then

$$K(\Gamma(X, \mathcal{A})) \simeq \Gamma(X, K(\mathcal{A})).$$

Sketch of the proof

Compare the two functors

$$CH/X \longrightarrow Mod(KU)$$

1) $(f : Y \to X) \longmapsto K(\Gamma(Y, f^*\mathcal{A}))$

2) $(f : Y \to X) \longmapsto \Gamma(Y, K(f^*\mathcal{A}))$

have a nat. transf.

constant sheaf $\to$ $\underline{\Gamma(Y, f^*\mathcal{A})} \Rightarrow \Phi_{f^*\mathcal{A}}$

given by restriction $\sigma^* : \Gamma(Y, f^*\mathcal{A}) \to \Gamma(\Delta^n, \sigma^* f^*\mathcal{A})$

Composing with K gives

$$\underline{K(\Gamma(Y, f^*\mathcal{A}))} \Rightarrow K \circ \Phi_{f^*\mathcal{A}}$$

$$\rightsquigarrow \quad K(\Gamma(Y, f^*\mathcal{A})) \overset{\rho_f}{\Rightarrow} \Gamma(Y, K(f^*\mathcal{A}))$$

-) needs some work to see that $f \mapsto \rho_f$ provides a nat. transf.
-) then reduce to case where X is fin. CW-complex and use cell-decomp. to show that ρ is an equiv. inductively.

Twisted K-theory and strongly self-abs. C^*-algebras

Observation: $KU = K(\mathbb{C})$ is a ring spectrum

space of units

$$\begin{tikzcd}
GL_1(KU) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^\infty KU \arrow[d] \\
GL_1(\pi_0(\Omega^\infty KU)) \arrow[r] & \pi_0(\Omega^\infty KU)
\end{tikzcd}$$

$GL_1(KU)$ is a commutative group

$gl_1(KU)$ in Spc.

$$[X, BGL_1(KU)]$$

Def.: Let $\mathcal{C}$ be a symm. mon. ∞-cat. A morphism $\varepsilon : 1\!\!1 \longrightarrow X$ is an *idempotent* in $\mathcal{C}$ if

$$\varepsilon \otimes id : X \simeq 1 \otimes X \xrightarrow{\varepsilon \otimes id} X \otimes X$$

is an equivalence.

Def.: A unital C^*-alg. A is called *strongly self-abs.* if $\exists$ iso $\varphi : A \xrightarrow{\cong} A \otimes A$ and a path $u : (0,1] \longrightarrow U(A \otimes A)$ s.th.

$$\lim_{t \to 0} \| u_t \phi(a) u_t^* - 1 \otimes a \| = 0$$

Remark: Since $u_t \phi(\cdot) u_t^*$ gives a path between the right tensor embedding and an iso.

Examples: $\mathbb{C}$, $\mathcal{O}_2$, $\mathcal{O}_\infty$, $M_n(\mathbb{C})^{\otimes \infty}$, $\mathcal{Z}$, tensor products

$\uparrow$ univ. C^*-alg. gen. by countably many partial isom. s_i s.th.

$$s_i^* s_j = \delta_{ij} \cdot 1$$

Lemma: If $\varepsilon : \mathbb{C} \longrightarrow A$ is an idempotent in C^*Alg_h, then $K(A)$ has a canonical structure of a comm. algebra in $Mod(KU)$.

Obs.: $\varepsilon : \mathbb{C} \longrightarrow A$ idempotent in C^*Alg_h

Then $L_A = \_ \otimes A$ is a Bousfield localisation

$\rightsquigarrow (L_A C^*Alg_h , A , \otimes)$ is a symm. mon. ∞-cat.

$\rightsquigarrow Map_{C^*Alg_h}(A, A)$ has

$$\left.\begin{array}{l} \cdot)\ \text{composition} \\ \cdot)\ \otimes\text{-product} \\ \quad (\text{comm. algebra str.}) \end{array}\right\} \begin{array}{l} \text{Eckmann-Hilton type} \\ \text{argument} \\ \rightsquigarrow \text{both agree up to} \\ \quad \text{equivalence} \end{array}$$

In particular we get

$$\begin{tikzcd} \ell\, \underline{End(A \otimes \mathbb{K})} \simeq Map_{C^*Alg_h}(A \otimes \mathbb{K}, A \otimes \mathbb{K}) \arrow[r, "KK"] & \Omega^\infty KK(A, A) \arrow[d, "\varepsilon^*"] \\ & \Omega^\infty K(A) \end{tikzcd}$$

$A \otimes \mathbb{K} \in C^*Alg$

restricting to hom. of comm. groups in Spc

$$\ell\, \underline{Aut(A \otimes \mathbb{K})} \longrightarrow gl_1(K(A))$$

Thm. (Dadarlat-P.): For $A = O_\infty$

$$\ell\, \underline{Aut(O_\infty \otimes \mathbb{K})} \simeq gl_1(K(O_\infty)) \simeq gl_1(KU)$$