Cyclic branched covers of alternating knots

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Outline

- Review of cyclic branched covers of knots.
- Cyclic branched covers as invariants.
- Statement of result.
- Peculiarities of alternating knots.
- Structure of the proof.
- Torus knot case.
- Hyperbolic knot case.

Review of cyclic branched covers of knots.

Construction (K a knot and n>1 an integer ⇝ $M(K,n)$ a closed 3-manifold.)

$\boldsymbol{S}^3 \setminus \mathcal{U}(K)$ admits a unique n-fold cyclic cover $M_n(K) \to \boldsymbol{S}^3 \setminus \mathcal{U}(K)$ induced by the kernel of the epimorphism $\pi_1(\boldsymbol{S}^3 \setminus \mathcal{U}(K)) \to H_1(\boldsymbol{S}^3 \setminus \mathcal{U}(K)) \to \mathbf{Z}/n\mathbf{Z}$.

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Definition

$M(K,n) = M_n(K) \cup \boldsymbol{S}^1 \times \boldsymbol{D}^1$, where the lift of a meridian of the knot K is identified to a meridian of the solid torus, is the (total space of the) n-fold cyclic branched cover of K.

Remark

The action on $M_n(K)$ of the group $\mathbf{Z}/n\mathbf{Z}$ of deck transformations extends to $M(K,n)$. The group fixes pointwise the core of the added solid torus.

This gives a quotient map $(M(K,n),\mathrm{Fix}(\mathbf{Z}/n\mathbf{Z})) \to (\boldsymbol{S}^3,K)$, the actual n-fold cyclic branched cover.

Review of cyclic branched covers of knots.

Construction (Rolfsen) (K a knot and $n>1$ an integer ⇝ $M(K,n)$ a closed 3-manifold.)

Cut open $\boldsymbol{S}^3$ along a Seifert surface Σ for K to obtain a manifold $\boldsymbol{S}^3 \setminus \mathcal{U}(\Sigma)$ with boundary $\Sigma_+ \cup \Sigma_-$.

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Definition

$M(K,n)$ the manifold obtained by gluing together n copies of $\boldsymbol{S}^3 \setminus \mathcal{U}(\Sigma)$ in such a way that the the boundary component of one copy is glued to, is the (total space of the) n-fold cyclic branched cover of K.

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Review of cyclic branched covers of knots.

K a knot and $n>1$ an integer ⇝ $M(K,n)$ a closed 3-manifold.

Definition

$M(K,n)$ is the (total space of the) n-fold cyclic branched cover of K.

Characterising property

$M(K,n)$ admits an orientation-preserving diffeomorphism φ of order n such that

- $\mathrm{Fix}(\varphi^k)=\boldsymbol{S}^1$ for all $0<k<n$,
- The space of orbits $(M(K,n),\mathrm{Fix}(\varphi))/\varphi$ is $(\boldsymbol{S}^3,K)$.

In particular there is a quotient map $(M(K,n),\mathrm{Fix}(\varphi)) \to (\boldsymbol{S}^3,K)$.

Review of cyclic branched covers of knots.

Example

K_0 the trivial knot, then $M(K_0,n)=\boldsymbol{S}^3$ for all $n>1$.

Take φ to be the standard n-rotation about a great circle.

Remark

It follows from the positive solution to Smith's conjecture that if $M(K,n)=\boldsymbol{S}^3$ then K is the trivial knot.

Cyclic branched covers as invariants.

Question

Cyclic branched covers are topological invariants of knots.

How good are they?

Cyclic branched covers as invariants.

Definition

Let K, K' be knots, and let $n>1$ be an integer.

If $M(K,n)=M(K',n)$ implies $K'=K$, then we say that K is determined by its n-fold cyclic branched cover.

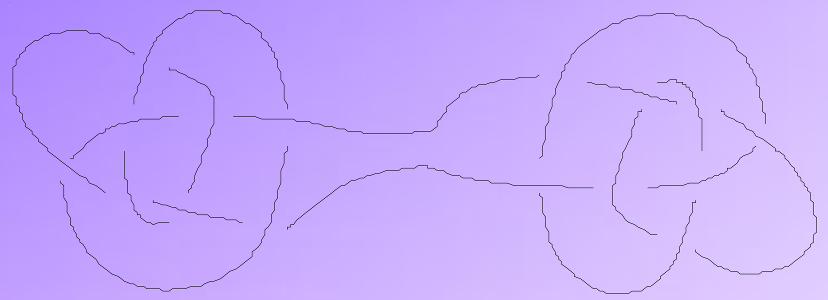
Else we say that $K'\neq K$ is an n-twin of K.

Cyclic branched covers as invariants.

Cyclic branched covers are very weak invariants of composite knots.

Example (Viro)

Two composite (alternating) knots, $8_{17}\#8_{17}$ and $8_{17}\#(-8_{17})$ that are n-twins for all n>1.



This construction generalises to give arbitrarily many composite knots that are n-twins for all n>1.

Cyclic branched covers as invariants.

Cyclic branched covers are stronger invariants of prime knots.

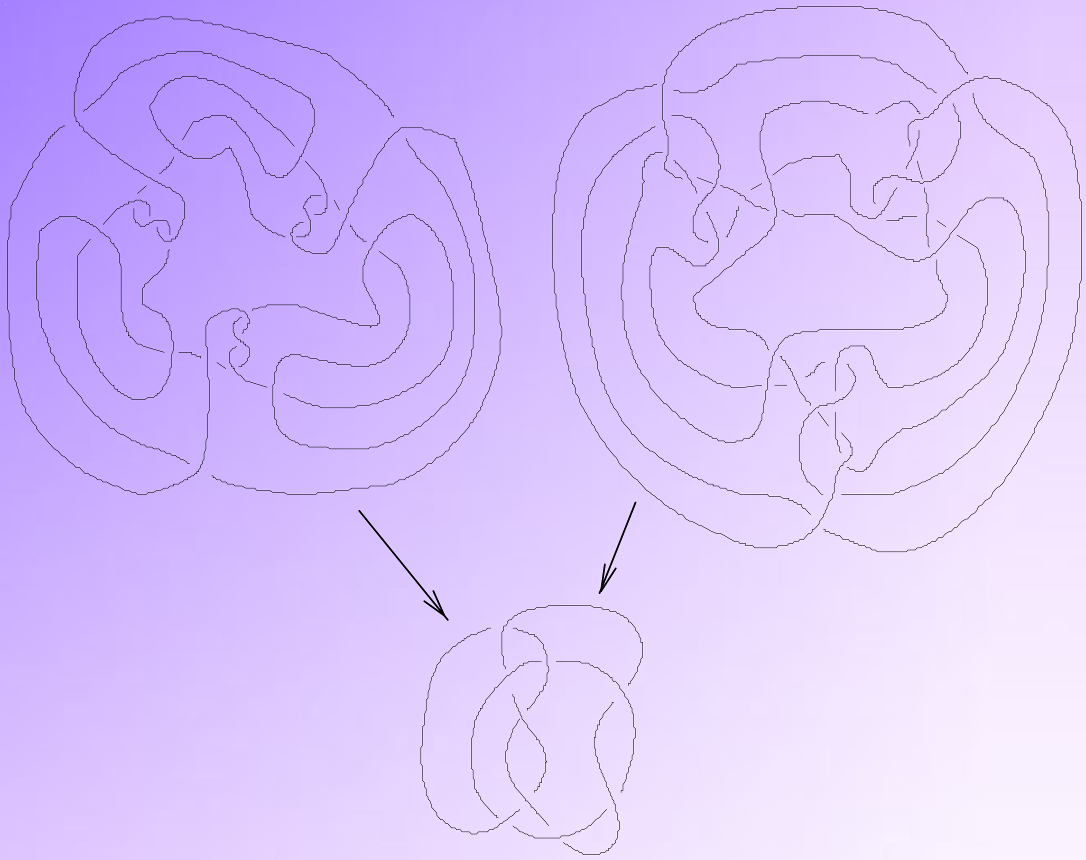
Theorem (Kojima)

For each prime knot K there is an integer $n(K)$ such that, two prime knots K and K' are equivalent if there exists an $n>\max(n(K),n(K'))$ for which $M(K,n)=M(K',n)$.

Cyclic branched covers as invariants.

Prime knots can have *n*-twins for *n* arbitrary large.

Example (Nakanishi and Sakuma's construction)



Cyclic branched covers as invariants.

Theorem (Zimmermann)

Let $n>2$. If K is a hyperbolic knot and K' an n-twin of K, then K and K' are obtained via Nakanishi and Sakuma's construction.

In particular, K has at most one n-twin, for $n>2$.

Remarks

- (Montesinos) There are hyperbolic knots with arbitrarily many 2-twins.
- (Zimmermann, P.) A knot K' can be a twin of a hyperbolic knot K for at most two integers $n>1$.

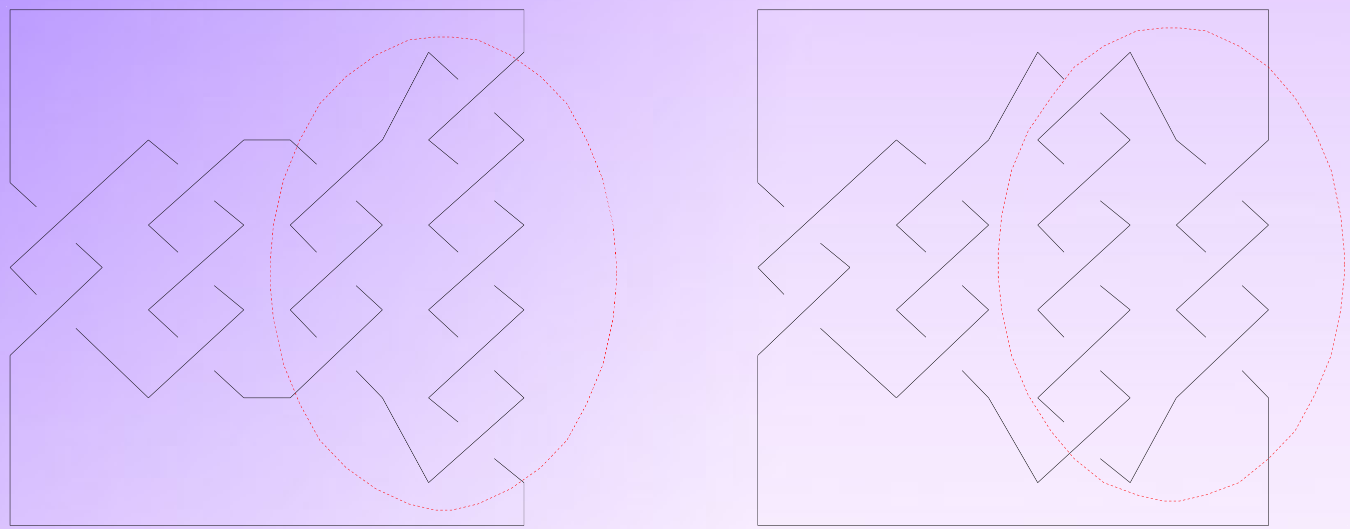
  (In general, only known for odd prime orders (Boileau-P.).)
- Toroidal prime knots may have n-twins, $n>2$, that are not obtained in this way (Boileau-P.).

Cyclic branched covers as invariants.

There are other constructions giving *n*-twins of prime knots, notably for *n*=2.

Example (Conway mutation: the case of Montesinos knots)

Two alternating pretzel knots obtained by Conway mutation. They are 2-twins.



Statement of result

Theorem (P.)

Let *K* be a prime knot and $n>2$. If *K* is alternating then it has no *n*-twins.

Statement of result

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Remark

The hypotheses are necessary.

Peculiarities of alternating knots.

Remark

According to Menasco, an alternating prime knot is atoroial, so it is either a torus knot or a hyperbolic knot (i.e. its exerior admits a complete hyperbolic structure of finite volume).

If it is a torus knot, it must be of type $(2,2k+1)$, for the others torus knots are not alternating according to Murasugi.

Peculiarities of alternating knots.

Remark

Costa and Quach Hongler proved that periods of order $m>2$ of prime alternating knots are visible on a minimal (alternating) diagram.

Their proof exploits Tait’s flyping conjecture as well as arguments of its proof by Menasco and Thistlethwaite.

Peculiarities of alternating knots.

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Definition

A period of order m of a knot K is an orientation-preserving diffeomorphism φ of order m of $\boldsymbol{S}^3$ such that

- $\varphi(K)=K$,
- $\mathrm{Fix}(\varphi^k)=\boldsymbol{S}^1$ for all $0<k<m$, (actually $k=1$ suffices)
- $\mathrm{Fix}(\varphi) \cap K = \emptyset$.

Peculiarities of alternating knots.

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Definition

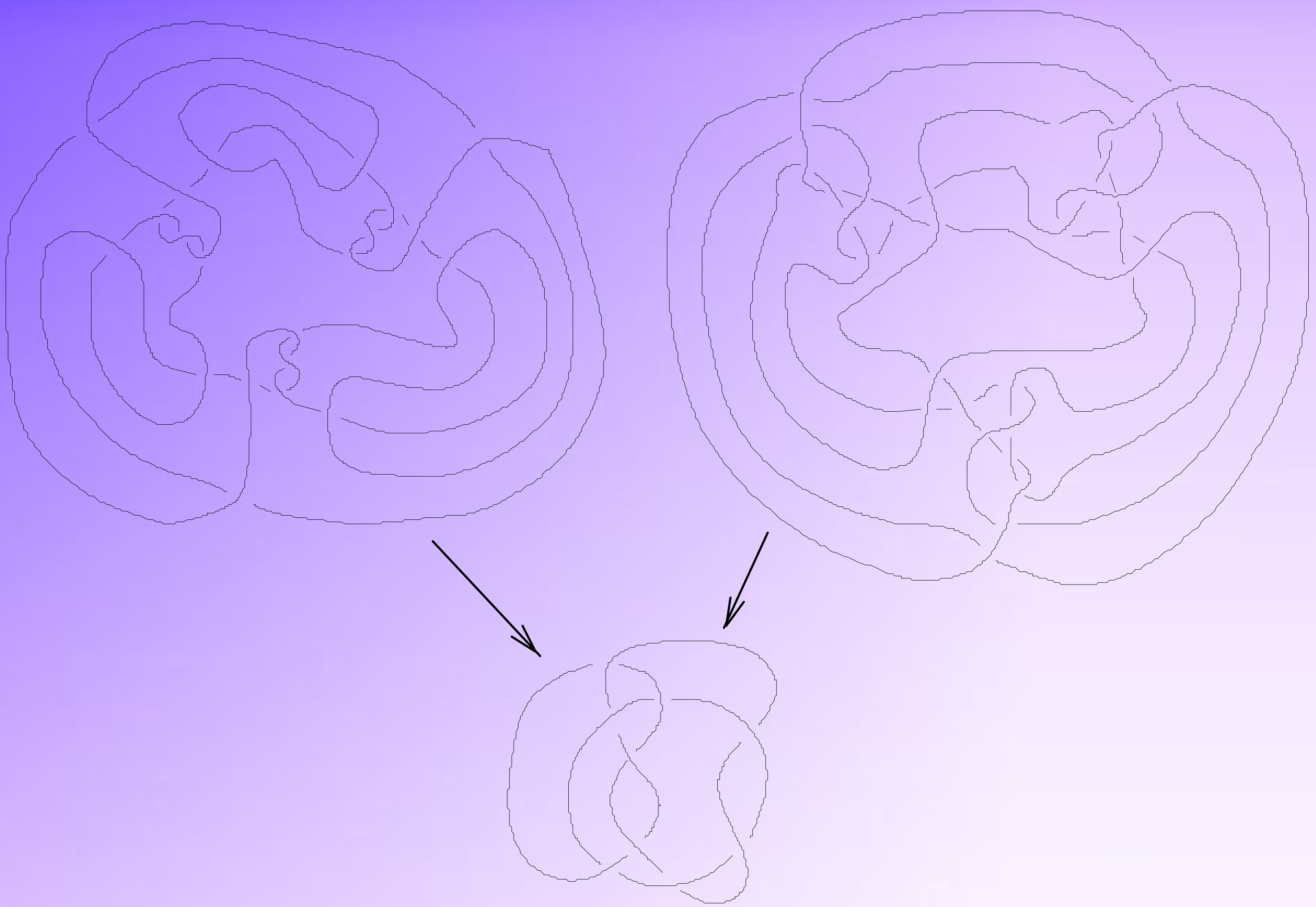
A period φ of order m is visible on a diagram D for K if there exist

- a 2-sphere S embedded in $\boldsymbol{S}^3$, such that $\varphi(S) = S$,
- two points, a, b $\in \mathrm{Fix}(\varphi)$, one on each side of S,
- a product structure $\boldsymbol{S}^2 \times (-1, 1)$ of $\boldsymbol{S}^3 \setminus \{a, b\}$, for which $S = \boldsymbol{S}^2 \times \{0\}$,

such that the projection $p : \boldsymbol{S}^2 \times (-1, 1) = \boldsymbol{S}^3 \setminus \{a, b\} \to \boldsymbol{S}^2 \times \{0\} = S$ satisfies $p(K) = D$, and there is a diffeomorphism $\psi: S \to S$ of order m such that $\psi \circ p = p \circ \varphi$.

Peculiarities of alternating knots.

Example



Structure of the proof.

- If K is composite, then $M(K,n)$ is not prime: composite knots can only have twins that are composite knots.

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- It follows from Thurston's orbifold theorem that if K is hyperbolic and $n>2$, then $M(K,n)$ is hyperbolic with a single exception, i.e. $n=3$ and K is the figure-eight knot. The figure-eight knot has no 3-twins, according to Dunbar's classification of geometric orbifolds.

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- If K is hyperbolic and $n=2$, anything goes! Hyperbolic knots may have 2-twins that are torus knots and even toroidal ones.

Structure of the proof.

If an alternating knot *K* has an *n*-twin for $n>2$, then the twin is of the same type as *K*.

We can consider the two cases, torus knots and hyperbolic knots, separately.

Torus knot case.

Proposition

Let $n>1$. Then two torus knots cannot be n-twins.

Proof

Follows from the classification of Brieskorn manifolds obtained by W. Neumann.

Corollary

Let $n>2$. Then a torus knot does not have n-twins.

Remark

Let $n>1$. Alternating torus knots have no n-twins (Hodgson-Rubinstein).

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Hyperbolic knot case.

Theorem (Zimmermann)

Let n>2. If two hyperbolic knots are n-twins, then they are obtained via Nakanishi and Sakuma's construction.

Equivalently:

Let K be a hyperbolic knot and n>2. K admits an n-twin iff K admits a period φ of order n such that

- the quotien knot K/φ is trivial,
- the components of the link $(K,\mathrm{Fix}(\varphi))/\varphi$ are not exchangeable.

Hyperbolic knot case.

The strategy is now to show the following:

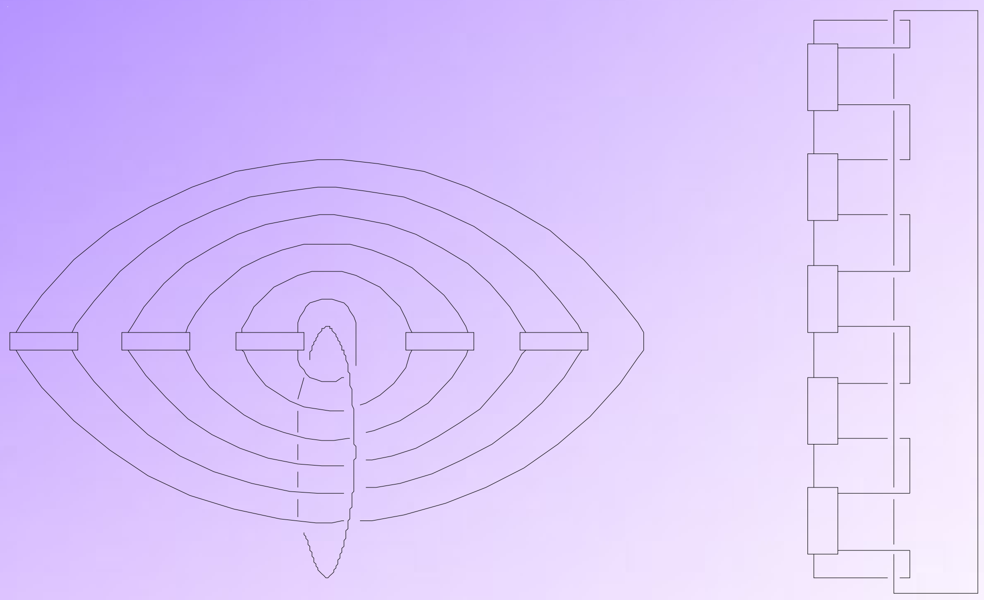
If an alternating knot admits a period with trivial quotient knot and the period is visible on a minimal (alternating) diagram, then the components of the corresponding quotient link are exchangeable.

Since periods of order >2 of prime alternating knots are visible, according to Zimmermann result, this will achieve the proof of the theorem.

Hyperbolic knot case.

Proposition

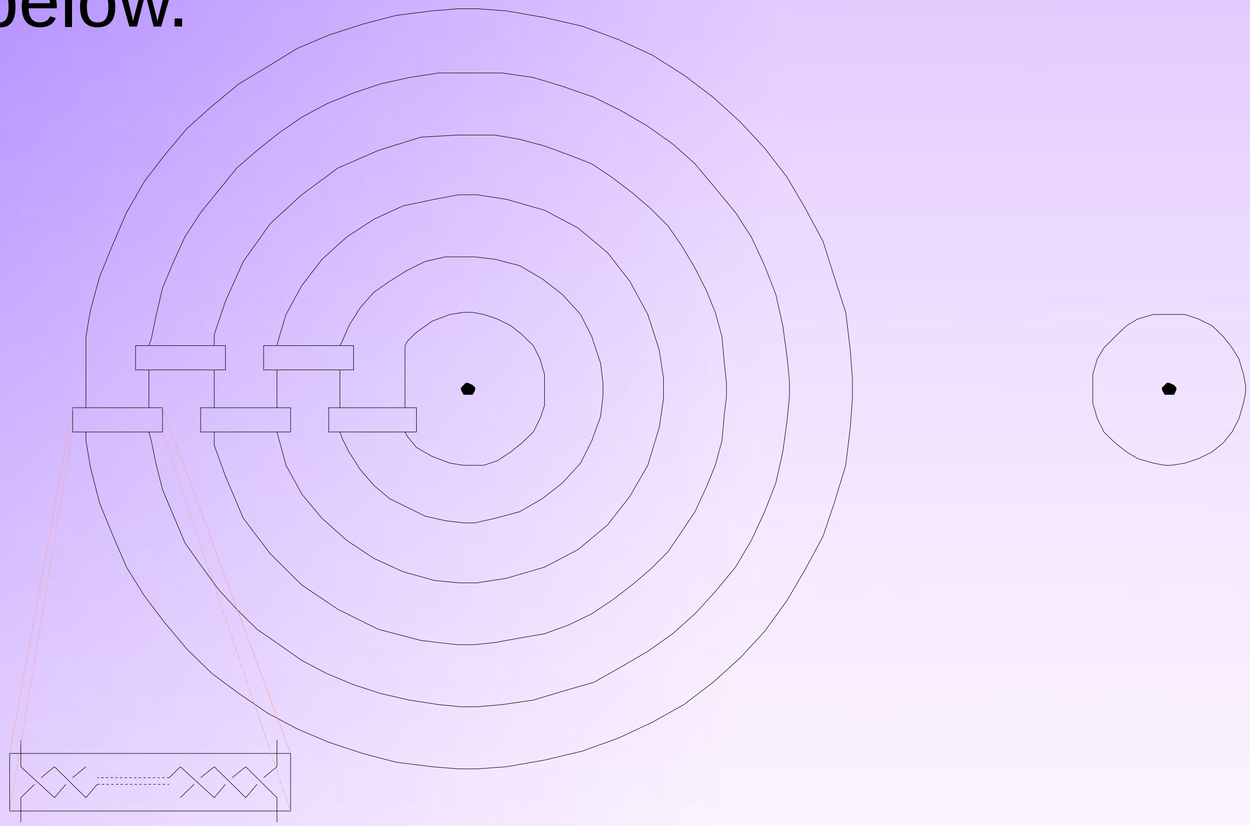
Let *K* be a prime alternating knot admitting an *n*-period *φ* such that *K*/*φ* is the trivial knot and *φ* is visible on a minimal diagram. Then (*K*,Fix(*φ*))/*φ* is a 2-bridge link of the form shown below, where boxes denote sequences of half-twists. In particular its two components are excheangable.



Hyperbolic knot case.

Lemma

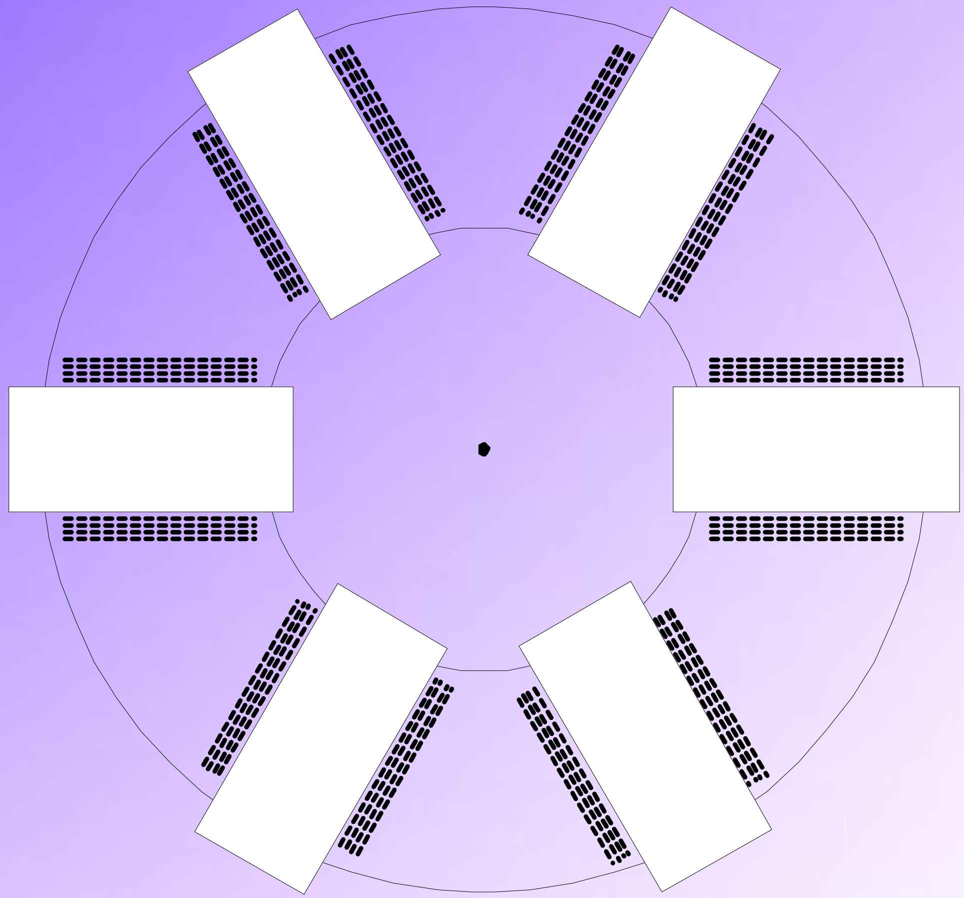
Under the hypotheses of the previous proposition, up to isotopy relative to Fix(φ)/φ, the trivial knot K/φ admits a diagram of the form below.



Hyperbolic knot case.

Proof

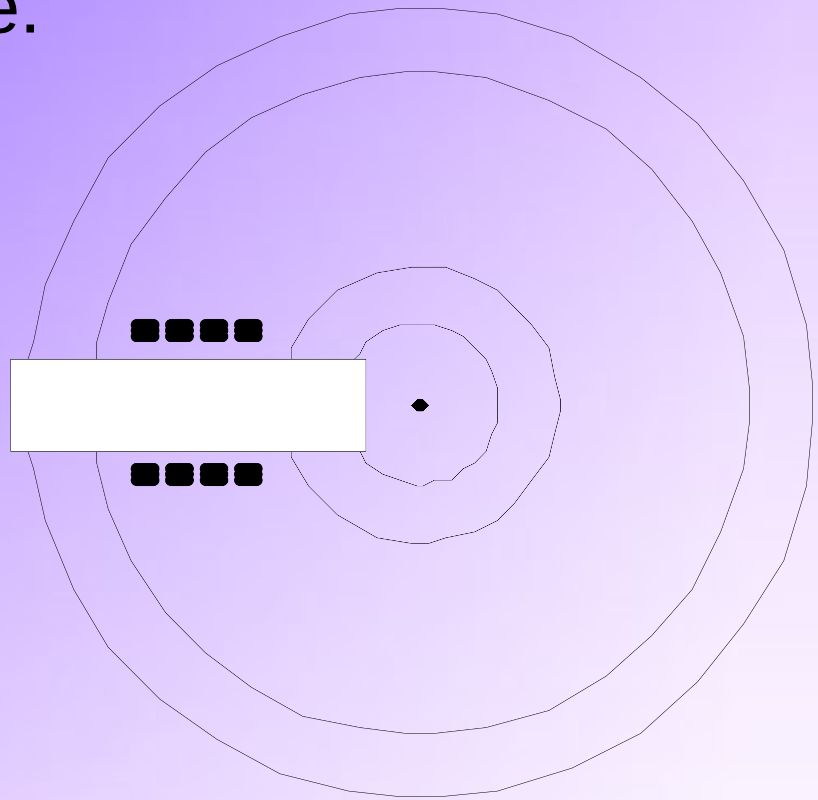
A schematic diagram with a visible period.



Hyperbolic knot case.

Proof

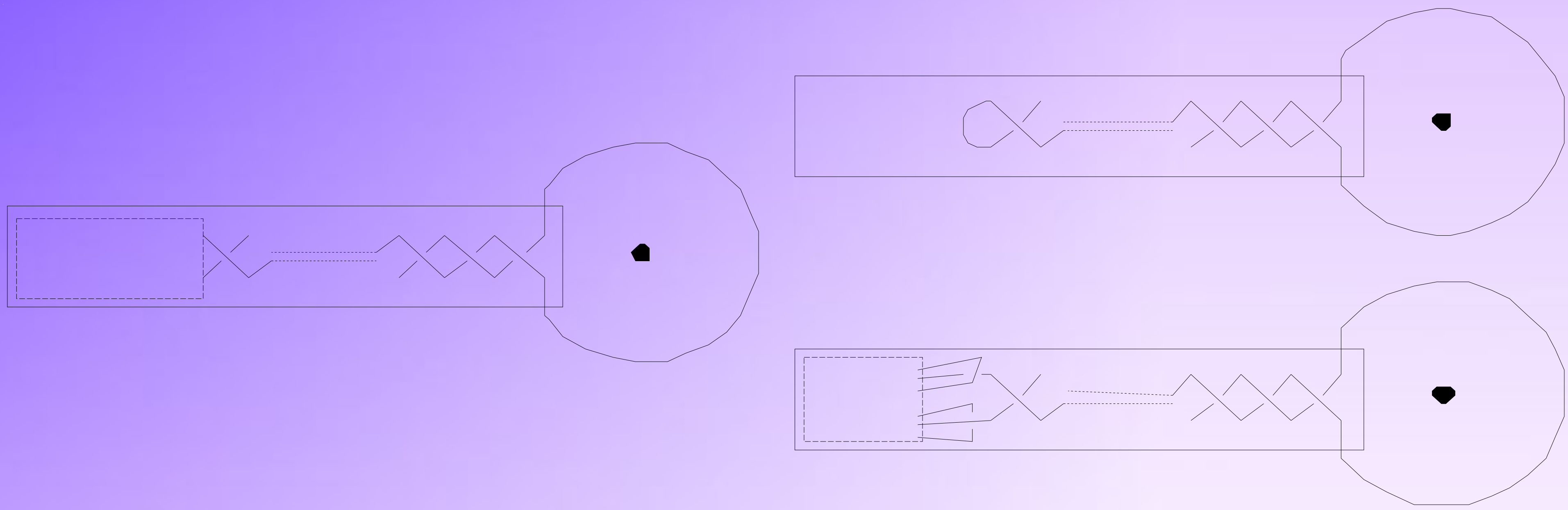
A quotient diagram. The proof is by induction on the size of the tangle, using the presence of a Reidemeister I move not included in the tangle.



Hyperbolic knot case.

Proof

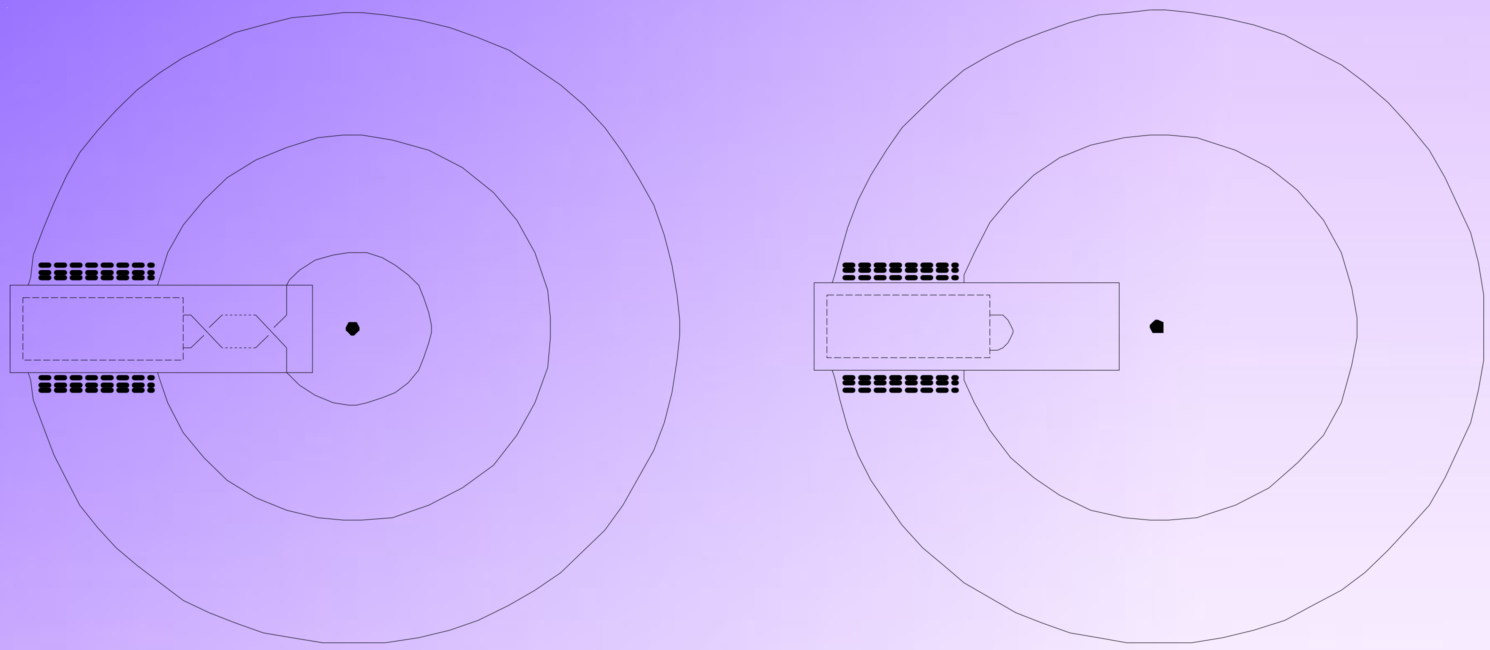
Base case.



Hyperbolic knot case.

Proof

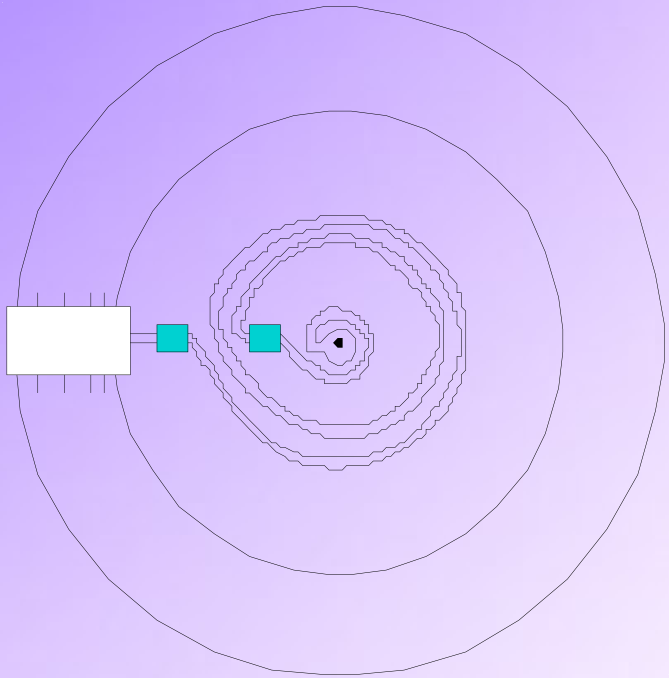
Induction step.



Hyperbolic knot case.

Proof

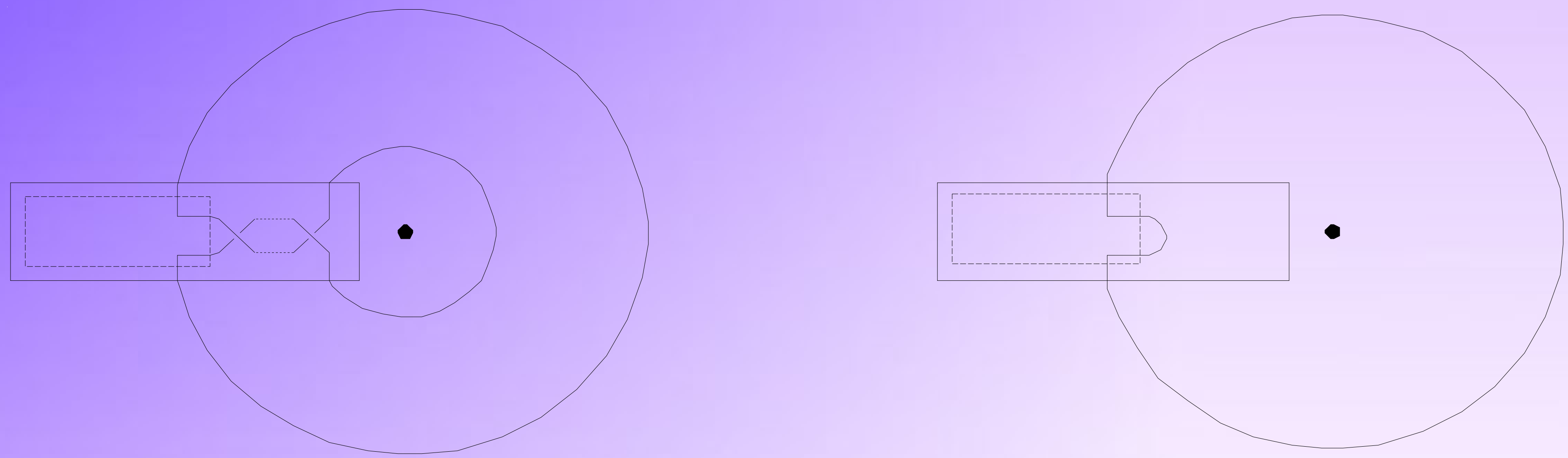
Induction step. Disregarding isotopy.



Hyperbolic knot case.

Proof

Induction step. No other crossings.



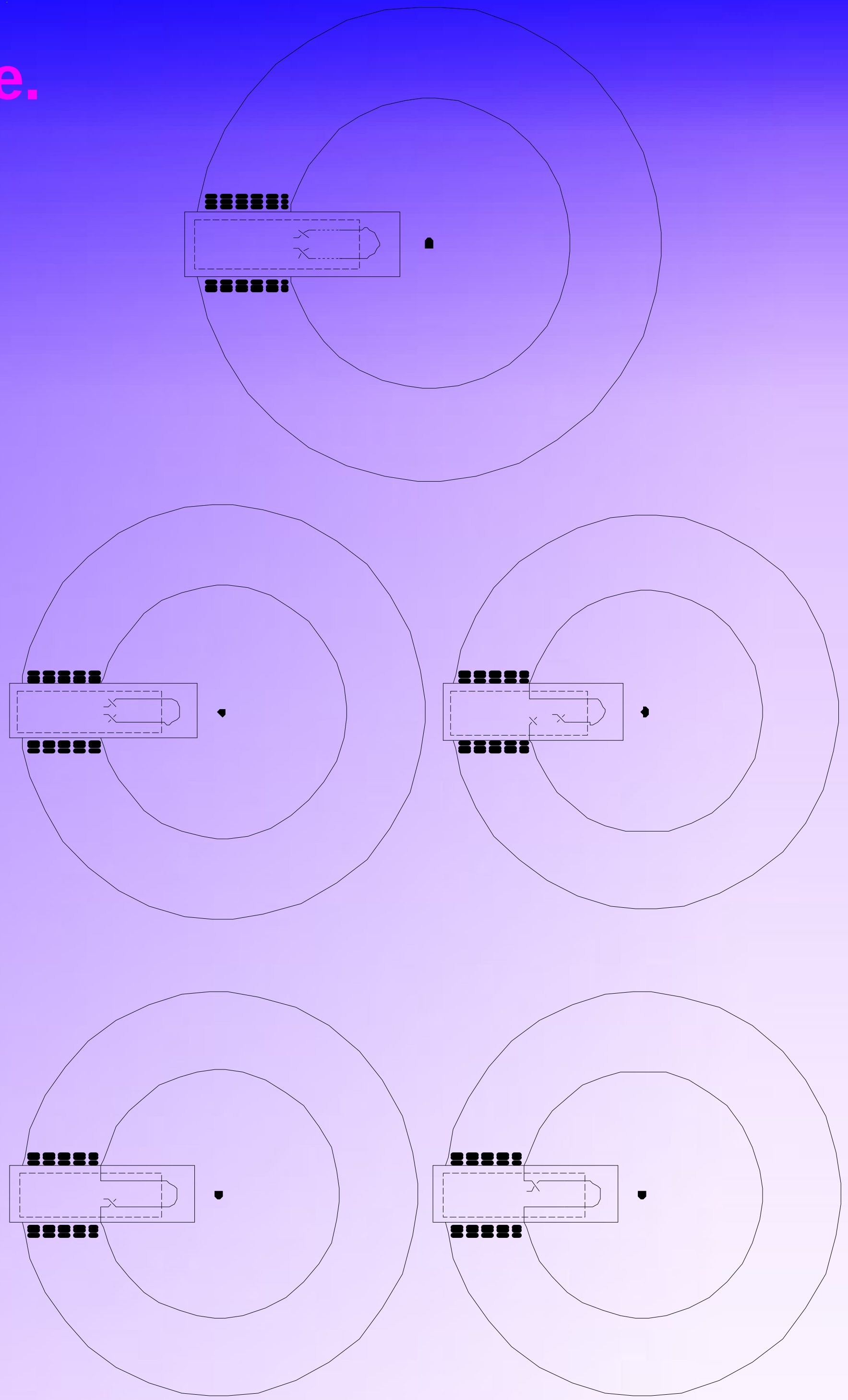
Hyperbolic knot case.

Proof

Induction step.

There are other

crossings.



Thank you for your attention