Heegaard genus, degree-one maps, and amalgamation of 3-manifolds

Monday, November 30, 2020 7:50 PM

Motivation.

Conj 1. Let M and N be closed ori. 3-mflds

Suppose $\exists$ a degree-one map $f: M \to N$

Then $g(M) \geq g(N)$, where $g(M)$ is the Heeg. genus.

Conj 1 $\Rightarrow$ Poincaré Conj. $M = S^3$

(Haken, Waldhausen) $f: M \to N = W_N \cup V_N$ H.S.

G = spine (of V_N)

after homotopy $f^{-1}(G) \cong G$

We can assume $V = f^{-1}(V_N) \cong V_N$

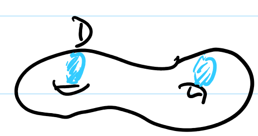
$M = W \cup_T V$

$f(W) = W_N$

$\underline{f(V) = V_N}$ $f|_V$ is a homeo.

Consider $f: W \to W_N$

$f|_{\partial W}$ is a homeo.



W_N

After homotopy $f^{-1}(D)$ is an incomp. surf. in W for any given ess. disk $D \subset W_N$

W ∂W = genus g surf.

W $\quad \partial W =$ genus g surf. 

g non-sep. s.c.c. in $\partial W = T$ $\quad \gamma_1, \cdots \gamma_g$

s.t. each γ_i bounds an incomp surf. F_i

Fact. $\exists$ a degree-one map $f: W \longrightarrow H$ (genus-g handlebody)
pinching each F_i to a disk.

Conj. 2. Let W & γ_i's be as above. and $M = W \cup_T V$
(V does not need to be a handlebody)
Let N be obtained by replacing W w/. a handlebody H
s.t. each γ_i bounds a disk in H.
Then $g(M) \geq g(N)$

$g = 1$, $T =$ torus $\quad W =$ exterior of a knot in a homology sphere

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Thm Let $M = W \cup_T V$, where W is the exterior of a knot in a homology sphere. Let $N = \hat{T} \cup_T V$ be obtained by replacing W w/. a solid torus $\hat{T}$ s.t ∂(Seifert surf) = meridian
Then $g(M) \geq g(N)$

Cor. The tunnel # of a Satellite knot $\geq$ tunnel # of its pattern knot.

pattern knot.

tunnels

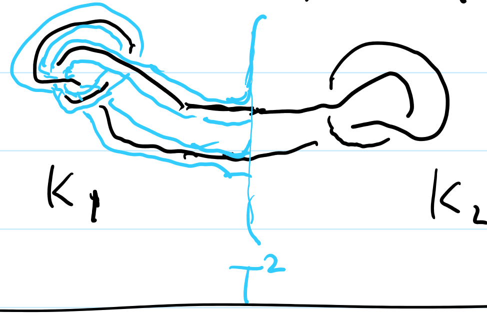
knot $\cup$ arcs $=$ graph G

unknotting tunnels if $S^3 \setminus G$ is a handlebody

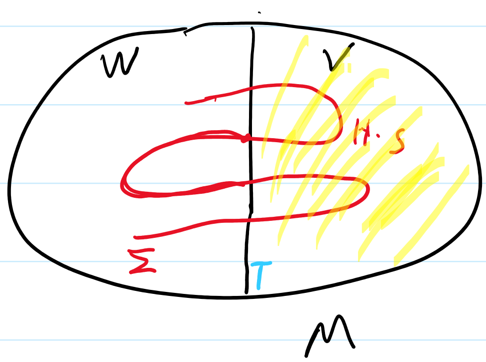
$\partial[N(G)]$ is a Heeg. surf. in $S^3 \setminus N(K)$

$$g(S^3 \setminus N(K)) = t(K) + 1$$

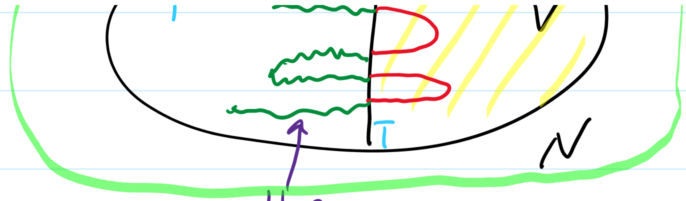
Cor (Schirmer 16) $t(K_1 \# K_2) \geq \max\{t(K_1), t(K_2)\}$



$M = W \cup_T V$



Need the Heeg. surf intersects W & V "nicely"



H.S. of N w/ same genus.

Fact. If T is incomp. and Σ is strongly irreducible. then $\Sigma \cap W$ and $\Sigma \cap V$ are "nice" after isotopy

Def. A H. S. $M = H_1 \cup_\Sigma H_2$ is reducible if $\exists$ an ess. simple closed curve in Σ that bounds disks in both H_1 H_2,

$$(S^2 \cap \Sigma = \text{a circle})$$

Lemma (Haken) If M is reducible, then every H. S. of M is reducible

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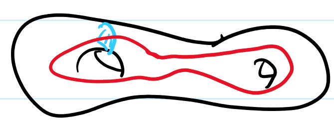
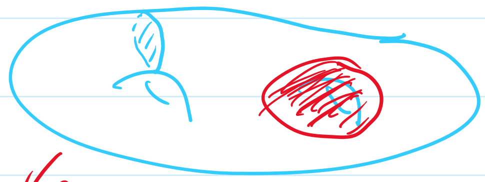
Def. A H. S. $M = H_1 \cup_\Sigma H_2$ is weakly reducible if $\exists$ ess. disks. $D_1 \subset H_1$, $D_2 \subset H_2$ s.t. $D_1 \cap D_2 = \phi$

If not weakly reducible, then it is strongly irred



Thm (Casson - Gordon) If M is non-Haken, then irred $\Longleftrightarrow$ strongly irred.

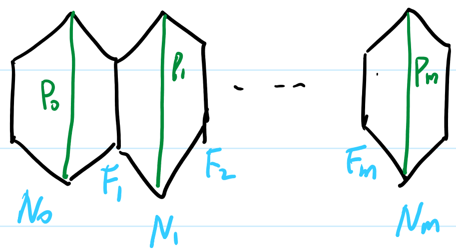


incomp. surf.

(Untelescoping)

Scharlemann - Thompson : for $\forall$ irred. Heeg. splitting

$\exists$ a decomp. of $M = N_0 \cup_{F_1} N_1 \cup_{F_2} \cdots \cup_{F_m} N_m$ along incomp.



surfaces $F_1, \cdots F_m$, Each N_i has a strongly irred H. S. P_i

Generalized Heeg. splitting.

For any H. S. $M = H_1 \cup_\Sigma H_2$

$$\Sigma = \partial \left(\{0\text{-handles}\} \cup \{1\text{-handles}\} \right)$$
$$= \partial \left(\{2\text{-handles}\} \cup \{3\text{-handles}\} \right)$$

rearrange these handles.

0-handle $\cup$ some 1-handles $\cup$ some 2-handles $\cup$ some 1-handle $\cup \cdots$

0-handle $\cup$ some 1-handles $\cup$ some 2-handles $\cup$ some 1-handle $\cup$ ······ −

$\partial = P_0$

$\partial = F_1$

$\partial = P_1$

$\partial = F_1$

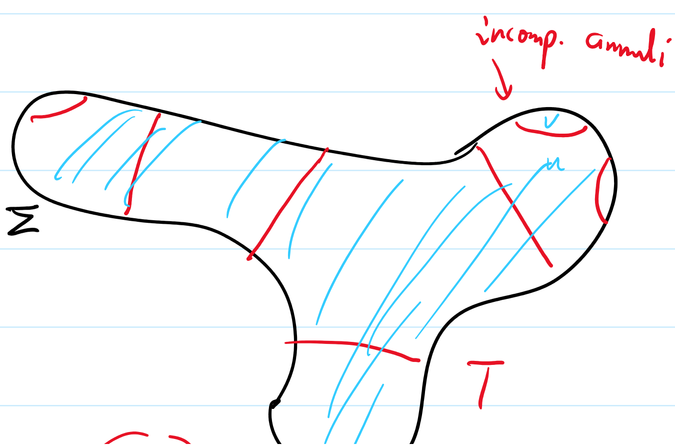
Lemma (Bachman – Schleimer – Sedgwick)

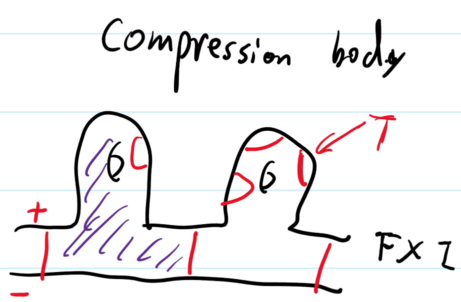
Let $M = W \cup_T V$ where T is an incomp. torus

Let Σ be the collection of incomp. & strongly irred. surfaces in the untelescoping. Then either

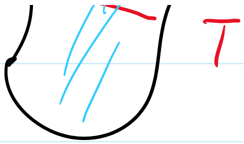
1) T is a component of some F_i or
2) each component of $\Sigma \cap W$ and $\Sigma \cap V$ is either incomp. or strongly irred in W, V resp.

Σ is str. irred





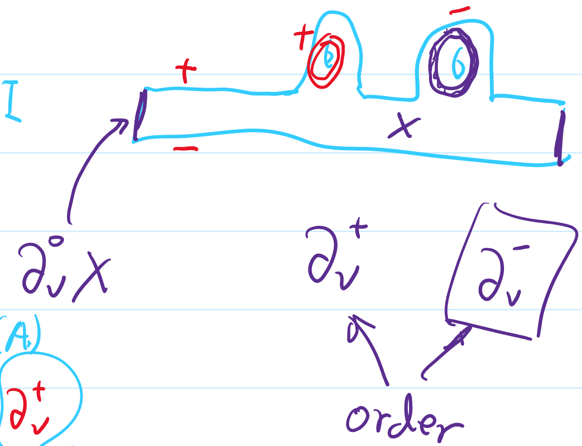
W

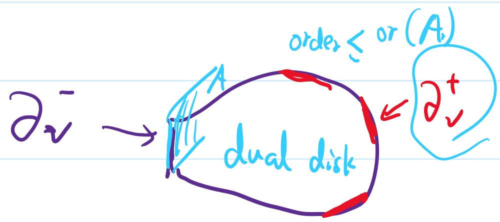


each piece is a relative compression body

F, $\partial F \neq \emptyset$

F×I

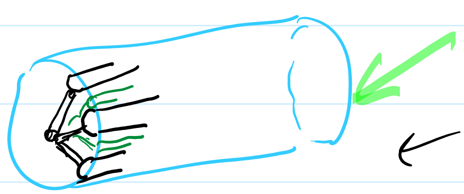




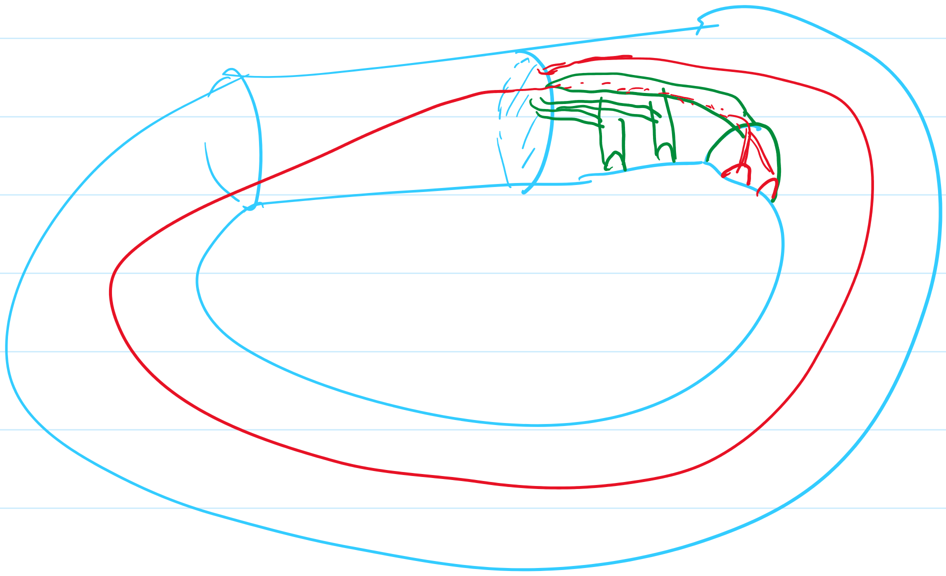
each annulus in $\partial_v^- X$ has a dual disk

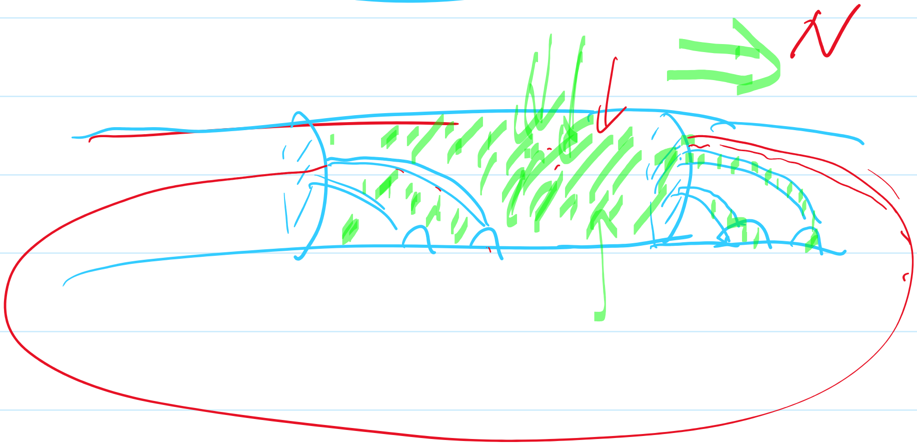
$\hat{T}$

Two cases i) each component in $\Sigma \cap W$ is separating.



ii) A component of $\Sigma \cap W$ is non-sep.




N


T
T^2