# Filtered instanton Floer homology and the 3-dimensional homology cobordism group
# (Joint work with Yuta Nozaki and Kouki Sato)

Masaki Taniguchi

RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program

Regensburg low-dimensional geometry and topology seminar, Nov. 10 2020

This slide is available at "https://sites.google.com/view/masaki-taniguchis-homepage".

## Contents

# Two cobordism groups

Let $Y_0$ and $Y_1$ be oriented homology 3-spheres. [1]

**Definition (The homology cobordism group)**

We say $Y_0$ is *homology cobordant* to $Y_1$ ($Y_0 \sim_{\mathbb{Z}H} Y_1$) if there exists a compact oriented smooth 4-manifold $W$ with $\partial W = Y_0 \amalg (-Y_1)$ such that the maps $H_*(Y_i, \mathbb{Z}) \to H_*(W, \mathbb{Z})$ induced by inclusions $Y_i \to W$ are isomorphisms.

$$\Theta^3_{\mathbb{Z}} := \{ \text{ oriented homology 3-spheres } \} / \sim_{\mathbb{Z}H}$$

Let $K_0$ and $K_1$ be oriented knots in $S^3$.

**Definition (The knot concordance group $\mathcal{C}$)**

We say $K_0$ is *concordant* to $K_1$ ($K_0 \sim_c K_1$) if there exists a smooth embedding $J : S^1 \times [0,1] \to S^3 \times [0,1]$ such that $J|_{S^1 \times \{i\}} = K_i \times \{i\}$ for $i = 0$ and $1$.

$$\mathcal{C} := \{ \text{ all oriented knots } \} / \sim_c$$

[1] A closed 3-manifold $Y$ is called a homology 3-sphere if $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$.

# Known results related to $\Theta^3_{\mathbb{Z}}$ and $\mathcal{C}$

- The rational homology cobordism group $\Theta^3_{\mathbb{Q}}$ is defined by replacing $\mathbb{Z}$ with $\mathbb{Q}$ in the definition of $\Theta^3_{\mathbb{Z}}$. The double branched cover gives a homomorphism
$$\Sigma : \mathcal{C} \to \Theta^3_{\mathbb{Q}}.$$
- 1969, Kervaire, the $n$-dimensional PL homology cobordism group $\Theta^n_{\mathbb{Z}}(PL)$ is trivial for $n \neq 3$. Moreover, $\Theta^3_{\mathbb{Z}}(PL) \cong \Theta^3_{\mathbb{Z}}$. $\exists$ similar classification result for higher dimensional knot concordance group (1969, Levine)
- 1976, Galewski–Stern, Matumoto, any topological manifold $M$ with $\dim \geq 5$ admits a triangulation $\iff$ $0 = \exists\delta(\Delta(M)) \in H^5(M, \text{Ker } \mu)$, where $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}_2$ is the Roklin homomorphism.
- 1982, Donaldson, Theorem A implies that $\Sigma(2,3,5)$ is not a torsion in $\Theta^3_{\mathbb{Z}}$. (YM gauge theory)
- 1990, Fintushel–Stern, Furuta, $\{\Sigma(p,q,pqn-1)\}_{n=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$. (YM gauge theory)

- 2002, Frøyshov, a surjective homomorphism

$$h : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$$

such that $h(\Sigma(2,3,5)) = 1 \Longrightarrow$ existence of a $\mathbb{Z}$-summand. (Floer homology in YM gauge theory)

- 2016, Manolescu, Roklin homomorphism $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}_2$ does not split $\Longrightarrow$ disproof of triangulation conjecture for $n \geq 5$. ($Pin(2)$-Seibeg-Witten Floer homology)
- 2018, Dai-Hom-Stoffregen-Truong, a surjective homomorphism

$$\Theta^3_{\mathbb{Z}} \to \mathbb{Z}^\infty.$$

$\Longrightarrow$ existence of a $\mathbb{Z}^\infty$-summand. (Involutive Heegaard Floer homology)

- 2018, Daemi, a family of real-valued functions parametrized by $k \in \mathbb{Z}$

$$\Gamma_Y(k) : \Theta^3_{\mathbb{Z}} \to [0, \infty]$$

(Floer homology in YM gauge theory)

- 2020, Nov, Hendricks-Hom-Stoffregen-Zemke, the manifold $S^3_1(-2T(6,7)\#T(6,13)\#T(-2,3;2,5))$ is not contained in the subgroup $\Theta^3_{SF}$ generated by Seifert homology 3-spheres. (Involutive Heegaard Floer homology) ( $T(-2,3;2,5)$ : $(2,5)$-cable of $T(-2,3)$, $2$ is the longitudinal winding. )

## Open questions on $\Theta^3_{\mathbb{Z}}$ and $\mathcal{C}$

The first point is that $-S_{1/n}(T_{p,q}) = \Sigma(p, q, pqn-1)$.

**Open question on $\Theta^3_{\mathbb{Z}}$**

Is there a nice sufficient condition of $K$ such that $\{S_{1/n}(K)\}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$?

The second point is a geometric structure of homology 3-spheres. We confirmed that Hendricks-Hom-Stoffregen-Zemke's example $S^3_1(-2T(6,7)\#T(6,13)\#T(-2,3;2,5))$ is a graph homology 3-sphere.

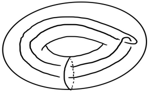
**Open question on $\Theta^3_{\mathbb{Z}}$**

Is $\Theta^3_{\mathbb{Z}}$ generated by all graph homology 3-spheres?

The Whitehead double[2] determines a map $D\colon \mathcal{C} \to \mathcal{C}$.

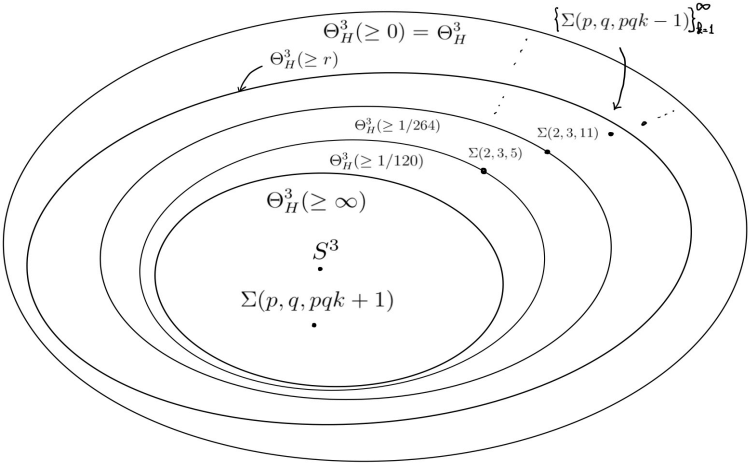
**Hedden–Kirk's conjecture**

The map $D$ preserves the linear independence.



## Idea to answer open questions

We will give a filtration of subgroups of $\Theta^3_{\mathbb{Z}}$ by giving a real-valued homology cobordism invariant.



## Main result

**Theorem (2019, Nozaki–Sato–T, (Floer homology in YM gauge theory))**

For $s \in \mathbb{R}_{\leq 0} \amalg \{-\infty\}$ and an oriented homology sphere $Y$, we define $r_s(Y) \in \mathbb{R}_{>0} \amalg \{\infty\}$ satisfying the following conditions:

1. (Monotonicity) If $s \leq s'$, then $r_{s'}(Y) \leq r_s(Y)$.
2. (Values) The values of $r_s(Y)$ are contained in the set of critical values of the $SU(2)$-Chern–Simons functional of $Y$.
3. (Negative Definite Inequality) Let $Y_0$ and $Y_1$ be $\mathbb{Z}$HS$^3$'s and $W$ a negative definite cobordism with $\partial W = Y_0 \amalg -Y_1$. Then $r_s(Y_1) \leq r_s(Y_0)$ holds for any $s$. If $\pi_1(W) = 1$ and $r_s(Y_0) < \infty$, then $r_s(Y_1) < r_s(Y_0)$ holds.
4. (Connectes Sum Inequality) The invariant $r_s$ satisfies
$$r_s(Y_1 \# Y_2) \geq \min\{r_{s_1}(Y_1) + s_2, r_{s_2}(Y_2) + s_1\}$$
for $s = s_1 + s_2 \in (-\infty, 0]$.
5. (Non-Triviality) $r_{-\infty}(Y) < \infty \iff h(Y) < 0$, where $h \colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}$ is the Frøyshov homomorphism.

## Remarks on the main theorem

- Daemi's invariants $\Gamma_Y(k)$ also satisfy the conditions $2$, $3$ and $5$ for a positive $k$. $\exists$a relation between $r_s(Y)$ and $\Gamma_Y(k)$? We proved that, for any oriented homology 3-sphere $Y$, $r_{-\infty}(-Y) = \Gamma_Y(1)$ (NST19).
- $\exists$ an example of $Y$ such that $r_s(Y)$ is not constant w.r.t. $s$.
- Roughly speaking, $r_0(Y)$ is given by

$$\inf\left\{ -\frac{1}{8\pi^2}\int_{Y\times\mathbb{R}} \mathrm{Tr}(F(A)\wedge F(A)) \;\middle|\; A \in \Omega^1_{Y\times\mathbb{R}}\otimes\mathfrak{su}(2) \text{ with } (*)\right\}$$

$$= \inf\left\{ cs(b) \;\middle|\; A \in \Omega^1_{Y\times\mathbb{R}}\otimes\mathfrak{su}(2) \text{ with } (*), b = \exists \lim_{t\to-\infty} A|_{Y\times\{t\}}\right\}$$

The conditions $(*)$ are given as follows:

- $0 = \exists \lim_{t\to\infty} A|_{Y\times\{t\}}$.
- $\exists$ Riemann metric $g$ on $Y$ such that the ASD-equation $\frac{1}{2}(1 + *_{g+dt^2})F(A) = 0$ is satisfied.
- The Fredholm index of the operator $d^+_A + d^*_A$ on $Y\times\mathbb{R}$ is $1$.

# Calculations

## Example

$r_s(S^3) = \infty$ for any $s$.

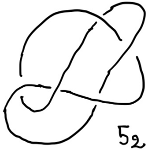
## Theorem (D18,NST19)

$\Gamma_{\Sigma(p,q,pqn-1)}(1) = r_s(-\Sigma(p,q,pqn-1)) = \frac{1}{4pq(pqn-1)}$ for any $s$.

In general,

$$\bigcup_s r_s(\Theta^3_{GR}) \subset \mathbb{Q}_{>0} \amalg \{\infty\},$$

where $\Theta^3_{GR}$ is the subgroup of $\Theta^3_{\mathbb{Z}}$ generated by graph homology 3-spheres. We tried to calculate $r_s$ for the hyperbolic manifold $S^3_{1/2}(5_2^*)$ obtained by the $1/2$-surgery along the mirror image of $5_2$.



$5_2$

# Calculations

## Theorem (NST, 19)

By computer, for any $s$,

$$r_s(S^3_{1/2}(5_2^*)) \approx 0.0017648904\ 7864885113\ 0739625897\ 0947779330\ 4925308209$$

whose error is $10^{-50}$, where $S^3_{1/2}(5_2^*)$ is the $1/2$ surgery on the mirror image of $5_2$ in Rolfsen's table.

Our computation is based on Kirk and Klassen's formula (to be explained in the next slide).

## Our conjecture

$r_s(S^3_{1/2}(5_2^*))$ is irrational.

If the conjecture is true, we can conclude that $\Theta^3_{\mathbb{Z}}/\Theta^3_{GR}$ is non-trivial.

# Computation of $r_s(S^3_{1/2}(5_2^*))$

Let $\rho_0$, $\rho_1$ be $SU(2)$-representations of $\pi_1 = \pi_1(S^3_{-1/2}(5_2))$ and $\{\rho_s\}_s \subset \mathrm{Hom}(\pi_1(S^3 \setminus 5_2), SL(2,\mathbb{C}))$ a path from $\rho_0$ to $\rho_1$.
Then Kirk and Klassen gave a fomula of the form

$$cs(\rho_1) - cs(\rho_0) \equiv \int_0^1 \text{“}\rho_s(\lambda) \ \&\ \rho_s(\mu)\text{”} ds \mod \mathbb{Z}.$$

The irreducible representations of $\pi_1(S^3 \setminus 5_2)$ are described by the Riley polynomial
$\phi(t,u) = -(t^{-2}+t^2)u + (t^{-1}+t)(2+3u+2u^2) - (3+6u+3u^2+u^3)$.

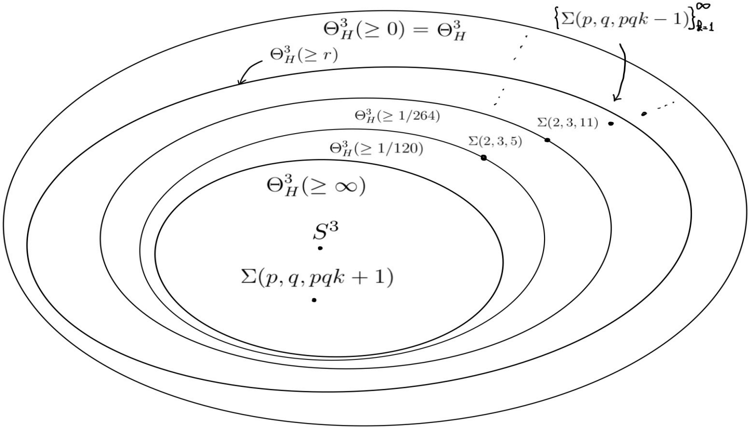
| | $t$ | $u$ | $-cs$ |
|---|---|---|---|
| $\rho_1$ | $0.716932 + 0.697143i$ | $-0.0755806$ | $0.00176489$ |
| $\rho_2$ | $0.309017 + 0.951057i$ | $-1.00000$ | $0.166667$ |
| $\rho_3$ | $-0.339570 + 0.940581i$ | $-2.41421$ | $0.604167$ |
| $\rho_4$ | $-0.778407 + 0.627759i$ | $-1.69110$ | $0.388460$ |
| $\rho_5$ | $-0.809017 + 0.587785i$ | $-1.00000$ | $0.166667$ |
| $\rho_6$ | $-0.905371 + 0.424621i$ | $-2.16991$ | $0.865934$ |
| $\rho_7$ | $-0.912712 + 0.408603i$ | $-3.62043$ | $0.321158$ |
| $\rho_8$ | $-0.988857 + 0.148870i$ | $-2.41421$ | $0.604167$ |

We define

$$\Theta^3_{\mathbb{Z}}(\geq r) := \{[Y] \in \Theta^3_{\mathbb{Z}} | \min\{r_0(Y), r_0(-Y)\} \geq r\}$$

for $r \in [0, \infty]$. We see that $\Theta^3_{\mathbb{Z}}(\geq r)$ is a subgroup because of the connected sum inequality

$$r_0(Y_1 \# Y_2) \geq \min\{r_0(Y), r_0(-Y)\}.$$



## Three applications of $\{r_s\}$

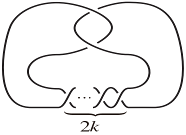
**Theorem ((I), NST, 19)**

For any knot $K$ in $S^3$ with $h(S_1(K)) < 0$, $\{S^3_{1/n}(K)\}$ are linearly independent in $\Theta^3_{\mathbb{Z}}$.

If we take $K = T_{p,q}$, this theorem recovers the result of Furuta, Fintushel–Stern in '90.

**Proposition**

All positive $k$-twisted knots ($k \geq 1$) and $(2, q)$-cable knots ($q \geq 3$) satisfy $h(S_1(K)) < 0$.



# Useful lemmas

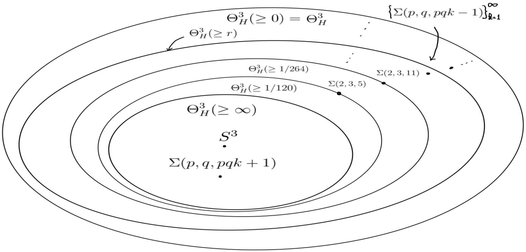
In the proof of Theorem (I), we use the following property of $r_0$.

**Lemma**

Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of oriented homology 3-spheres satisfying the following two conditions:

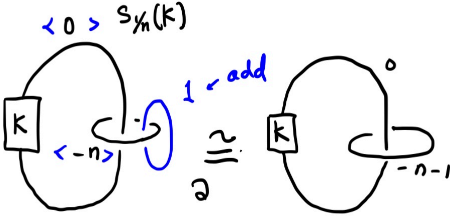
- $r_0(Y_1) > r_0(Y_2) > \cdots$ and
- $r_0(-Y_n) = \infty$ for any $n$.

Then the sequence $\{[Y_n]\}$ are linearly independent in both $\Theta^3_{\mathbb{Z}}$ and $\Theta^3_{\mathbb{Q}}$.



# Sketch of the proof of the Theorem (I)

Set $Y_n := S_{1/n}(K)$. The Non-Triviality and Negative Deinite Inequality of $r_0$ implies $r_0(Y_1) < \infty$ and $r_0(-Y_n) = \infty$. On the other hand, we have a positive definite cobordism $W_n$ with $\partial(W_n) = -Y_n \amalg (Y_{n+1})$ described by




One can see that $W_n$ is simply connected for each $n$. Therefore the strict version of Negative Deinite Inequality of $r_0$ implies that

$$r_0(Y_1) > r_0(Y_2) > \cdots .$$

□

# Three applications of $\{r_s\}$

**Theorem ((II), NST, 19)**

$\exists$ infinitely many homology spheres $\{Y_k\}$ such that $Y_k$ does not admit any definite bounding.

Set $Y_k := 2\Sigma(2,3,5)\#(-\Sigma(2,3,6k+5))$. $(k \geq 1)$ Then using Connected Sum Inequality, we have $r_0(Y_k) = \frac{1}{24(6k+5)} < \infty$. Moreover, the calculation $h(-Y_k) = -1$ and Non-Triviality implies that $r_0(-Y_k) < \infty$. □

**Corollary (NST, 19)**

The class $[Y_k] \in \Theta^3_{\mathbb{Z}}$ does not contain any Seifert homology sphere and homology 3-sphere obtained by a surgery on a knot in $S^3$.

It is known that all Seifert homology spheres and homology 3-spheres obtained by surgeries on knots admit a definite bounding. □

Stoffregen('15) proved that $[\Sigma(2,3,11)\#\Sigma(2,3,11)]$ does not contain any Seifert homology 3-sphere. ($Pin(2)$-Seiberg-Witten Floer homology)

## Three applications of $\{r_s\}$

Let $T_{p,q}$ be the $(p,q)$-torus knot. In 2012, Hedden–Kirk proved that $\{D(T_{2,2^n-1})\}_{n=2}^{\infty}$ are lineary independent in $\mathcal{C}$.

**Theorem ((III), NST, 19)**

Let $(p,q)$ be a coprime pair. $\{D(T_{p,np+q})\}_{n=1}^{\infty}$ are lineary independent in $\mathcal{C}$.

Since

$$\Sigma\colon \mathcal{C} \to \Theta^3_{\mathbb{Q}}$$

is a homomorphism, it is sufficient to prove that $\{\Sigma(D(T_{p,kp+q}))\}_{k=1}^{\infty}$ are linearly independent in $\Theta^3_{\mathbb{Q}}$. Note that $\Sigma(D(T_{p,q})) = S^3_{1/2}(T_{p,q}\# T_{p,q})$ is $\mathbb{Z}HS^3$.

**Lemma**

- $r_0(\Sigma(D(T_{p,q})) < \infty$.
- $r_0(\Sigma(D(T_{p,q})) > r_0(\Sigma(D(T_{p,p+q})).$

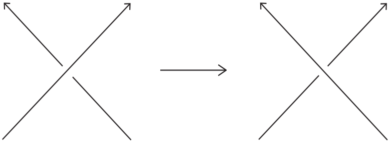
To prove the above lemma, we construct

- neg. defn. cob. with boundary $(-\Sigma(p,q,2pq-1)) \amalg (-\Sigma(D(T_{p,q}))$
- simp. conn. neg. defn. cob. with boundary $\Sigma(D(T_{p,q})) \amalg (-\Sigma(D(T_{p,p+q}))$

# Sketch of the proof of Theorem (III)

**Lemma**

If $K_0 \to \cdots \to K_1$ by a seq. of pos. crossing changes, then $\exists$ neg. defn. cob. with boundary $S^3_{1/n}(K_1) \amalg (-S^3_{1/n}(K_0))$ for $\forall n$.



,

- $T_{p,q} \# T_{p,q} \xrightarrow{\text{pos. c.c.}} T_{p,q} \rightsquigarrow r_0(\Sigma(D_{p,q})) < \infty$
- $T_{p,q+p} \# T_{p,q+p} \xrightarrow{\text{pos. c.c.}} T_{p,q} \# T_{p,q} \rightsquigarrow r_0(\Sigma(D_{p,q})) > r_0(\Sigma(D_{p,p+q}))$

□

# History of instanton homology related to our work

Let $Y$ be an oriented homology 3-sphere.

- 1987, Floer, Instanton homology $I_*(Y)$ with $* \in \mathbb{Z}/8\mathbb{Z}$.
- 1992, Fintushel–Stern, Filtered version of instanton homology $I_*^{[r,r+1]}(Y)$ with $* \in \mathbb{Z}$ for $r \in \mathbb{R}$.
- 2002, Donaldson, The obstruction class $[\theta_Y] \in I^1(Y)$. If $Y$ admits a negative definite bounding with non-standard intersection form, then $0 \neq [\theta_Y] \in I^1(Y;\mathbb{Q})$.
- 2019, NST, Filtered instanton cohomology $I^*_{[s,r]}(Y)$ and the filtered version $[\theta_Y^{[s,r]}] \in I^*_{[s,r]}(Y)$ of the obstruction class.

**Definition**

$r_s(Y) := \sup\{r \in \mathbb{R} \mid 0 = [\theta_Y^{[s,r]}] \in I^*_{[s,r]}(Y)\}$

Such a quantitative construction in Floer theory appears in several situations including Hamiltonian Floer homology and embedded contact homology.

## Construction of $I^*_{[s,r]}$ and $[\theta_Y^{[s,r]}]$

Let $Y$ be an oriented homology 3-sphere. Set $\mathcal{B}_Y := \Omega^1_Y \otimes \mathfrak{su}(2)/\mathrm{Map}^0(Y, SU(2))$, where $\mathrm{Map}^0(Y, SU(2))$ is the set of null-homotopic smooth maps and the action is given by $a * g := g^{-1}dg + g^{-1}ag$. The (perturbed) Chern–Simons functional

$$cs_h : \mathcal{B}_Y \to \mathbb{R}$$

is given by

$$cs([a]) := \frac{1}{8\pi^2} \int_Y \mathrm{Tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a) + h$$

for some perturbation $h \colon \mathcal{B}_Y \to \mathbb{R}$ . The "critical point set" of $cs_h$ is given by

$$R_h(Y) = \{[a] \in \mathcal{B}_Y \mid F(a) + * \operatorname{grad}_a h = 0\}.$$

Floer defined the Floer index

$$\mathrm{ind}_h \colon R_h(Y) \to \mathbb{Z}$$

under some good situation.

## Gradient flow of $cs$

Fix a Riemann metric on $Y$. We equip an $L^2$-inner product on $\Omega^1_Y \otimes \mathfrak{su}(2)$ by

$$(a, b) := -\frac{1}{4\pi^2}\int_Y \mathrm{Tr}(a \wedge *b).$$

Then the formal gradient flow of $cs$ w.r.t. the inner product is given by

$$\text{grad }(cs{+}h)\colon a \mapsto -*_g (F(a)) + \text{grad } h.$$

A downward gradient flow $c : \mathbb{R} \to \Omega^1_Y \otimes \mathfrak{su}(2)$ of grad $(cs{+}h)$ corresponds to a solution to the ASD-equation

$$\frac{1}{2}(1 + *_{g+dt^2})(F(A) + \pi_h(A)) = 0,$$

where $A$ is the $SU(2)$-connection on $Y \times \mathbb{R}$ given by $A|_{Y\times t} = c(t)$ such that ($dt$-component of $A$) $= 0$.

# In the case of $Y = -\Sigma(2,3,5)$

The critical point set is

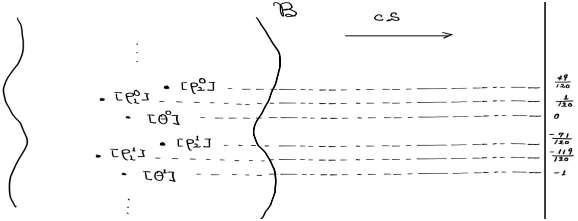
$$R(Y) = \{\rho_1^i, \rho_2^i, \theta^i\}_{i\in\mathbb{Z}}.$$

The critical values are

$$cs(\rho_1^i) = \frac{1}{120} + i,\ cs(\rho_2^i) = \frac{49}{120} + i \text{ and } cs(\theta^i) = i.$$

The Floer indicies are given by

$$\text{ind}(\rho_1^i) = 1 + 8i,\ \text{ind}(\rho_2^i) = 5 + 8i \text{ and } \text{ind}(\theta^i) = -3 + 8i.$$



# Construction of $I_*$

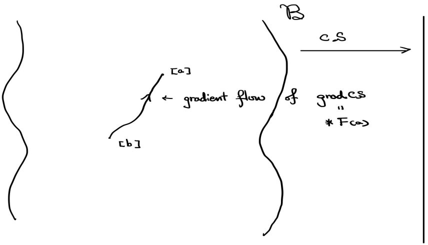
Suppose that $cs + h$ is Morse. ( $\iff$ Hess $(cs+h)_a$ : Ker$d_a^* \to$ Ker$d_a^*$ is injective for any critical point $a$.) The instanton Floer chain is given by

$$CI_*(Y) := \mathbb{Z}\{[a] \in R_h(Y) \setminus \{\theta^i\} \mid \text{ind}_h([a]) = *\}.$$

The differential is defined by

$$\partial([a]) = \sum_{[b]\in R(Y),\ \text{ind}([a])-\text{ind}([b])=1} \#(M_h([a],[b])/\mathbb{R})[b],$$

where the space $M_h([a],[b])$ is the set of trajectories of $cs+h$ from $[a]$ to $[b]$.



## Construction of $I_*$

When we give a topology on $M_h([a],[b])$, we use the identification

$$M_h([a],[b]) \cong \{A \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{k,loc}} \mid (*)\}/\mathcal{G},$$

where the conditions $(*)$ are given by

- $A - p^*a \in L^2_k(Y \times (-\infty,-1])$, $A - p^*b \in L^2_k(Y \times [1,\infty))$ and
- $(1 + *_{g+dt^2})(F(A) + \pi_h(A)) = 0$ (ASD equation),

where the map $p$ is the projection $Y \times \mathbb{R} \to Y$. The gauge group $\mathcal{G}$ is

$$\left\{ g \in \text{Map } (Y \times \mathbb{R}, SU(2))_{L^2_{k,loc}} \;\middle|\; \begin{array}{l} g^*p^*a \in L^2_k(Y \times (-\infty,-1]), \\ g^*p^*b \in L^2_k(Y \times [1,\infty)) \end{array} \right\}.$$

(One can check that the group $\mathcal{G}$ acts on the space $\{A \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{k,loc}} \mid (*)\}$.)

# Construction of $I_*$

## Theorem (Floer)

There exists a nice class of perturbations $h : \mathcal{B} \to \mathbb{R}$ of $cs$ satisfying the following conditions:

- The map $\partial$ is well-defined, i.e. , $M_h([a],[b])$ has a structure of a manifold of dimension $\text{ind}([a]) - \text{ind}([b])$ such that $\mathbb{R}$ action on $M_h([a],[b])$ is proper and free if $\text{ind}([a]) - \text{ind}([b]) > 0$ and $M_h([a],[b])/\mathbb{R}$ is compact if $\text{ind}([a]) - \text{ind}([b]) = 1$. Moreover, there is a method to give orientations on $M_h([a],[b])$.
- $\partial^2 = 0$ holds.
- The chain homotopy type of $(CI_*, \partial)$ does not depend on $h$ and $g_Y$.

The instanton (co) homology is given by $I_*(Y) := H_*(CI_*, \partial)$.

## Example

$$I_*(-\Sigma(2,3,5)) \cong \begin{cases} \mathbb{Z} \text{ if } * = 1, 5 \mod 8 \\ 0 \text{ otherwise} \end{cases}$$

# The obstruction class $[\theta]$

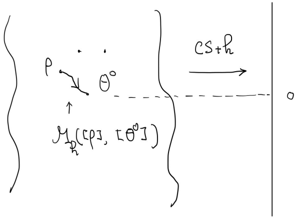
## Definition

The homomorphism $\theta : CI_1 \to \mathbb{Z}$ is given by $[a] \mapsto \#M_h([a], [\theta^0])$.

One can see that $\partial^*\theta = 0$. Therefore, the map $\theta$ determines a class $[\theta] \in I^1(Y)$. Although, the definition of the map $\theta$ depends on the choice of $h$ and $g_Y$, the cohomology class does not depend on the choices of $h$ and $g_Y$.

## Example

If $Y = -\Sigma(2, 3, 5)$, $\theta : CI_1 \to \mathbb{Z}$ satisfies $\theta(\rho_1^0) = \pm 1$. In this case, $[\theta]$ generates $I^1(Y)$.



# Construction of $I^*_{[s,r]}$

## Definition

For $s \in \mathbb{R}_{\leq 0} \setminus cs(R(Y)) \amalg \{-\infty\}$ and $r \in \mathbb{R}_{\geq 0} \setminus cs(R(Y))$, we define

$$CI_*^{[s,r]}(Y) := \mathbb{Z}\left\{ [a] \in R_h(Y) \setminus \{\theta^i\} \;\middle|\; \begin{array}{l} \mathrm{ind}([a]) = *, \\ s < (cs+h)([a]) < r \end{array} \right\}.$$

The differential $\partial^{[s,r]}$ is given by the restriction of $\partial$. The filtered instanton cohomology is given by

$$I^*_{[s,r]}(Y) := H_*(\mathrm{Hom}\,(CI_*^{[s,r]}(Y), \mathbb{Z}), (\partial^{[s,r]})^*).$$

## Theorem (Fintushel–Stern, '92)

If we take a small perturbation $h$ to define $I^*_{[s,r]}(Y)$, the chain homotopy type of $(\mathrm{Hom}\,(CI_*^{[s,r]}(Y), \mathbb{Z}), (\partial^{[s,r]})^*)$ does not depend on the choice of $h$ and $g_Y$.

## The obstruction class $[\theta^{[s,r]}]$

### Definition

For $s \in \mathbb{R}_{\leq 0} \setminus cs(R(Y)) \amalg \{-\infty\}$ and $r \in \mathbb{R}_{\geq 0} \setminus cs(R(Y))$, we have the homomorphism $\theta^{[s,r]} : CI_1^{[s,r]} \to \mathbb{Z}$ given by $[a] \mapsto \#M_h([a],[\theta^0])$.

One can see that $(\partial^{[s,r]})^*\theta = 0$. Therefore, the map $\theta^{[s,r]}$ determines a class $[\theta^{[s,r]}] \in I^1_{[s,r]}(Y)$. Moreover, for a small perturbation $h$, the class $[\theta^{[s,r]}] \in I^1_{[s,r]}(Y)$ is well-defined.

### Example

Suppose that $Y = -\Sigma(2,3,5)$.

- If $0 < r < \frac{1}{120}$, then the map $\theta^{[s,r]} : CI_1^{[s,r]} \to \mathbb{Z}$ is the zero map since $CI_1^{[s,r]} = 0$.
- If $\frac{1}{120} < r$, then the map $\theta^{[s,r]} : CI_1^{[s,r]} \to \mathbb{Z}$ gives an isomorphism.

# Definition of $r_s$

## Definition

For a given oriented homology 3-sphere $Y$ and $s \in [-\infty, 0] \setminus cs(R(Y))$,

$$r_s(Y) := \sup\{r \mid 0 = [\theta^{[s,r]} \otimes \mathrm{Id}_{\mathbb{Q}}] \in I^1_{[s,r]}(Y;\mathbb{Q})\}.$$

When $s \in cs(R(Y))$, we define

$$r_s(Y) := \lim_{t \to s-0} r_t(Y).$$

## Example

Suppose that $Y = -\Sigma(2,3,5)$.

- If $0 < r < \frac{1}{120}$, then $0 = [\theta^{[s,r]}] \in I^1_{[s,r]}$.
- If $\frac{1}{120} < r$, then $0 \neq [\theta^{[s,r]}] \in I^1_{[s,r]}$.

Therefore, $r_s(-\Sigma(2,3,5)) = \frac{1}{120}$.

## Negative definite inequality of $\{r_s\}$

For a negative definite cobordism $W$ with $\partial W = Y_0 \amalg (-Y_1)$ and $H_1(W, \mathbb{R}) = 0$, $s \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$ and $r \in \mathbb{R}_{\geq 0} \setminus (cs(R(Y_0)) \cup cs(R(Y_1)))$ , we have the cobordism map

$$CW : I^*_{[s,r]}(Y_1; \mathbb{Q}) \to I^*_{[s,r]}(Y_0; \mathbb{Q})$$

with $CW(\theta^{[s,r]}_{Y_1}) = c(W)\theta^{[s,r]}_{Y_0}$, where $c(W)$ is a non-zero rational number. This map is defined by counting the solutions to the ASD-moduli space for $W$. This gives the inequality

$$r_s(Y_0) \leq r_s(Y_1).$$

Moreover, If $r_s(Y_1) < \infty$ and $r_s(Y_0) = r_s(Y_1)$, one can construct an irreducible $SU(2)$-representation of $\pi_1(W)$. Therefore, if $\pi_1(W) = 1$ and $r_s(Y_1) < \infty$, we have

$$r_s(Y_0) < r_s(Y_1).$$

# Cobordism inequality of $\{r_s\}$

To prove $r_0(Y_0\#Y_1) \geq \min\{r_0(Y_0), r_0(Y_1)\}$, we need to show if $[\theta^{[0,r]}_{Y_i}] = 0$ for $i = 0$ and $1$ then $[\theta^{[0,r]}_{Y_0\#Y_1}] = 0$. Let $W$ be a cobordism with $\partial W = Y_0\#Y_1 \amalg (-Y_0) \amalg (-Y_1)$ obtained by adding a 3-handle on $Y_0\#Y_1$. There are four kinds of maps on the instanton chain complex induced by $W$;

- $p_0CW : CI^{[0,r]}_*(Y_0\#Y_1) \to CI^{[0,r]}_*(Y_0) \otimes CI^{[0,r]}_*(Y_1)$,
- $p_1CW : CI^{[0,r]}_*(Y_0\#Y_1) \to CI^{[0,r]}_*(Y_1)$,
- $p_2CW : CI^{[0,r]}_*(Y_0\#Y_1) \to CI^{[0,r]}_*(Y_0)$ and
- $p_3CW : CI^{[0,r]}_*(Y_0\#Y_1) \to \mathbb{Q}$.

Moreover, these maps satisfy nice equations related to $[\theta^{[0,r]}_{Y_0}]$, $[\theta^{[0,r]}_{Y_1}]$ and $[\theta^{[0,r]}_{Y_0\#Y_1}]$. Using such equations and the assumption $[\theta^{[0,r]}_{Y_i}] = 0$, one can see $[\theta^{[0,r]}_{Y_0\#Y_1}] = 0$.

# Further directions

- Recently, Daemi-Scaduto '19 constructed $\Gamma_K(k)$ for knots using quantitative instanton knot Floer homology. $\exists$ $r_s$-type invariants? $\exists$ connected sum formula?
- $\exists$ a local equivalence theory of quantitative instanton theory? Daemi, Sato and I are discussing now. (As an application, we gave connected sum inequalities for $\Gamma_Y(k)$.) Daemi-Scaduto '19 gave a formulation of local equivalence in quantitative instanton knot Floer homology.
- Can we prove $\Theta^3_{\mathbb{Z}}(\geq \infty)$ contain $\mathbb{Z}^\infty$ as a subgroup? $\{\Sigma(p, q, pqk + 1)\}$ are linearly independent?
- In my recent paper [arXiv:1910.02234], I gave a relation between Seifert hypersurfaces of smooth 2-knots and irreducible $SU(2)$-representations of their knot groups as applications of $\Gamma_Y(k)$ and $r_s(Y)$. (The difference between smooth and topological 2-knots.)

Thank you! Any comments are welcome!